Geometric Quantization of Real Minimal Nilpotent Orbits.

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Abstract: In this paper, we begin a quantization program for nilpotent orbits $O_{\mathbb{R}}$ of a real semisimple Lie group $G_{\mathbb{R}}$. These orbits arise naturally as the coadjoint orbits of $G_{\mathbb{R}}$ which are stable under scaling, and thus they have a canonical symplectic structure ω where the $G_{\mathbb{R}}$-action is Hamiltonian. These orbits and their covers generalize the oscillator phase space $T^*\mathbb{R}^n$, which occurs here when $G_{\mathbb{R}} = Sp(2n, \mathbb{R})$ and $O_{\mathbb{R}}$ is minimal.

A complex structure $\mathbf{J}$ polarizing $O_{\mathbb{R}}$ and invariant under a maximal compact subgroup $K_{\mathbb{R}}$ of $G_{\mathbb{R}}$ is provided by the Kronheimer-Vergne Kaehler structure $(\mathbf{J}, \omega)$. We argue that the Kaehler potential serves as the Hamiltonian. Using this setup, we realize the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ of $G_{\mathbb{R}}$ as a Lie algebra of rational functions on the holomorphic cotangent bundle T^*Y where $Y = (O_{\mathbb{R}}, \mathbf{J})$.

Thus we transform the quantization problem on $O_{\mathbb{R}}$ into a quantization problem on T^*Y. We explain this in detail and solve the new quantization problem on T^*Y in a uniform manner for minimal nilpotent orbits in the non-Hermitian case. The Hilbert space of quantization consists of holomorphic half-forms on Y. We construct the reproducing kernel. The Lie algebra $\mathfrak{g}_{\mathbb{R}}$ acts by explicit pseudo-differential operators on half-forms where the energy operator quantizing the Hamiltonian is inverted. The Lie algebra representation exponentiates to give a minimal unitary ladder representation of a cover of $G_{\mathbb{R}}$. Jordan algebras play a key role in the geometry and the quantization.

§1. Introduction.

I. Quantization of Phase Space. Quantization of a classical phase space M with symplectic form ω is a process whereby observables ϕ are converted into self-adjoint operators $\mathcal{Q}(\phi)$ on a Hilbert space $\mathcal{H}$ of states. The observables are simply the smooth functions on M.

The Hilbert space $\mathcal{H}$ should arise, according to the philosophy of Geometric Quantization, as a space of polarized sections of a suitable complex line bundle over M. A real (complex) polarization of M consists of a integrable Lagrangian distribution inside the (complexified) tangent bundle. A polarized section, of a bundle with connection, is a section annihilated by all vector fields lying in the polarization; in the real case, this means that the section is covariantly constant along the leaves of the corresponding Lagrangian foliation.

We require that the quantization satisfies Dirac's axioms (see e.g., [Ki], [A-M]) in some form. Dirac's consistency axiom is that the Poisson bracket of functions on M goes over into the commutator of operators so that

$$\mathcal{Q}(\{\phi, \psi\}) = i[\mathcal{Q}(\phi), \mathcal{Q}(\psi)] \tag{1.1}$$

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(We have set $\hbar = 1$.) Additional axioms mandate that the constant function 1 quantizes to the identity operator, and a complete set of observables quantizes to give a complete set of operators.

In Hamiltonian mechanics, the physics of the system in encoded in a single observable F (usually written as E or H) called the *Hamiltonian.* Often F is the total energy.

Any observable ϕ generates a Hamiltonian flow: this is the flow of the Hamiltonian vector field ξ_ϕ defined by the equation

$$\xi_\phi \lrcorner \omega + \mathrm{d}\phi = 0 \tag{1.2}$$

The Poisson bracket on $C^\infty(M)$ is given by $\{\phi, \psi\} = \xi_\phi(\psi) = \omega(\xi_\phi, \xi_\psi)$.

The Hamiltonian flow of the F gives the time evolution of the physical system. For any observable ϕ, the time derivative $\dot{\phi}$ of ϕ as the system evolves is given by $\dot{\phi} = \{F, \phi\}$. This is a concise version of Hamilton's equations. On physical grounds, in certain circumstances, F should be a positive function on M.

In quantization of a Hamiltonian mechanical system, F should be promoted to a self-adjoint operator $\mathcal{Q}(F)$ on $\mathcal{H}$ with positive spectrum. When F is the total classical energy, the spectrum of $\mathcal{Q}(F)$ should be discrete and give the possible quantized energy levels of the quantum system.

II. Quantization of the n-dimensional Harmonic Oscillator. The most familiar model situation is the case where M is the cotangent bundle of some (configuration) manifold X and ω is the canonical symplectic form so that $\omega = \mathrm{d}\theta$ where θ is the Liouville 1-form on T^*X. In this case we have the manifest cotangent polarization where the leaves are the cotangent spaces of X . We expect $\mathcal{H}$ to be a space of square integrable half-forms on X (see §2 and below starting around (1.5)). A smooth function f on X quantizes to a give a multiplication operator on $\mathcal{H}$. If η is a vector field on X, then the symbol σ^η quantizes to the Lie derivative $\mathcal{L}_\eta$ operator on half-forms. Consistent quantization of additional observables is problematic, as we see already in the oscillator example below.

A second model situation is the case where M is a Kaehler manifold and ω is the Kaehler form. Then the complex structure $\mathbf{J}$ of M gives a complex polarization. Now "polarized" simply means "holomorphic". Thus $\mathcal{H}$ should be a space of holomorphic square-integrable sections of a suitable holomorphic complex line bundle over M.

The most familiar example of a Hamiltonian mechanical system, the oscillator phase space, admits both cotangent and Kaehler polarizations. The oscillator phase space is $M = T^*\mathbb{R}^n$. The canonical coordinates on $T^*\mathbb{R}^n$ are the position coordinates $q_1, \ldots, q_n$ together with the momentum coordinates $p_1, \ldots, p_n$. The canonical symplectic form is $\omega = \sum_{k=1}^n \mathrm{d}p_k \wedge \mathrm{d}q_k$. The Poisson bracket satisfies $\{p_j, p_k\} = \{q_j, q_k\} = 0$ and $\{p_j, q_k\} = \delta_{jk}$. For general observables we have the classical formula

$$\{\phi, \psi\} = \sum_{k=1}^n \left(\frac{\partial \phi}{\partial p_k} \frac{\partial \psi}{\partial q_k} - \frac{\partial \psi}{\partial p_k} \frac{\partial \phi}{\partial q_k} \right) \tag{1.3}$$

In physics, $T^*\mathbb{R}^n$ arises as the phase space of n uncoupled harmonic oscillators with Hamil-

tonian equal to the total energy (kinetic plus potential)

$$F = \frac{1}{2}\sum_{k=1}^{n}(p_k^2 + q_k^2) \tag{1.4}$$

We also have a natural Kaehler structure. We identify $T^*\mathbb{R}^n = \mathbb{R}^{2n} = \mathbb{C}^n$ so that the complex-valued observables $z_k = (p_k + iq_k)/\sqrt{2}$ are holomorphic coordinates. Now $\mathbb{C}^n$ is a Kaehler manifold with Kaehler form ω and Kaehler metric $g = \sum_{k=1}^{n}(\mathrm{d}p_k^2 + \mathrm{d}q_k^2)$. In the $z_j, \overline{z}_k$ coordinates we have $\omega = i\sum_{k=1}^{n} \mathrm{d}\overline{z}_k \wedge \mathrm{d}z_k$ and the Poisson bracket satisfies $\{z_j, z_k\} = \{\overline{z}_j, \overline{z}_k\} = 0$ and $\{\overline{z}_j, z_k\} = i\delta_{jk}$. Also (1.4) becomes

$$F = \sum_{k=1}^{n} |z_k|^2 \tag{1.5}$$

The quantization of the Kaehler phase space $M = \mathbb{C}^n$ gives the Fock-Bargmann model of the quantum mechanical oscillator. (Quantization by means of the real cotangent polarization gives the Schroedinger model.) In this model, $\mathcal{H}$ is a space of holomorphic functions $f(z_1, \ldots, z_n)$ on $\mathbb{C}^n$. The Hamiltonian F quantizes into the energy operator

$$\mathcal{Q}(F) = \sum_{k=1}^{n}\left(z_k\frac{\partial}{\partial z_k} + \frac{1}{2}\right) \tag{1.6}$$

The functions z_k and $\overline{z}_k$ quantize into the creation and annihilation operators

$$\mathcal{Q}(z_k) = z_k \qquad \text{and} \qquad \mathcal{Q}(\overline{z}_k) = \frac{\partial}{\partial z_k} \tag{1.7}$$

Then $\mathcal{Q}(F)$ is a grading operator on the quantum space and $\mathcal{Q}(z_k) = z_k$ and $\mathcal{Q}(\overline{z}_k)$ are raising and lowering operators moving the eigenspaces of $\mathcal{Q}(F)$.

One way to "explain" the $\frac{1}{2}$-shift in (1.6) (a quantum correction) is to adopt the symmetrization procedure of canonical quantization so that

$$\mathcal{Q}(z_k\overline{z}_k) = \frac{1}{2}\left(\mathcal{Q}(z_k)\mathcal{Q}(\overline{z}_k) + \mathcal{Q}(\overline{z}_k)\mathcal{Q}(z_k)\right) = \frac{1}{2}\left(z_k\frac{\partial}{\partial z_k} + \frac{\partial}{\partial z_k}z_k\right) = z_k\frac{\partial}{\partial z_k} + \frac{1}{2} \tag{1.8}$$

There is a unique Hermitian inner product $\langle f|g\rangle$ on the space $H = \mathbb{C}[z_1, \ldots, z_n]$ of polynomial functions such that the operators $\mathcal{Q}(z_k)$ and $\mathcal{Q}(\overline{z}_k)$ in (1.7) are mutually adjoint. (The condition that $\mathcal{Q}(\phi)$ is self-adjoint for real ϕ amounts to the condition that $\mathcal{Q}(\phi)$ and $\mathcal{Q}(\overline{\phi})$ are mutually adjoint for complex ϕ.) This inner product is positive definite with

$$\|z_1^{a_1}\cdots z_m^{a_m}\|^2 = a_1!\cdots a_m! \tag{1.9}$$

The inner product (1.9) is given by the integral formula

$$\langle f|g\rangle = \int_{\mathbb{C}^m} f(z)\overline{g(z)}\mathrm{e}^{-|z|^2}|\mathrm{d}z\mathrm{d}\overline{z}| \tag{1.10}$$

and this expression defines the inner product on the Hilbert space completion $\mathcal{H}$ of H. Thus $\mathcal{H}$ consists of all the holomorphic functions $f(z_1, \ldots, z_n)$ on $\mathbb{C}^n$ which are "square integrable" in the sense that $\|f\|^2 = \langle f|f\rangle$ is finite.

The reproducing kernel (see §8) of $\mathcal{H}$ is the holomorphic function $\mathcal{K}(z, \overline{w})$ on $\mathbb{C}^n \times \overline{\mathbb{C}^n}$

$$\mathcal{K}(z, \overline{w}) = \exp(z_1\overline{w}_1 + \cdots + z_n\overline{w}_n) \tag{1.11}$$

Here $\overline{X}$ denotes the complex conjugate manifold to a complex manifold X, so that holomorphic functions on $\overline{X}$ identify with anti-holomorphic functions on X. $\overline{X}$ is obtained from X by reversing the sign of the complex structure.

The Hamiltonian flow of F lies inside a larger symmetry. The Hamiltonian F sits inside the space $\mathfrak{g}$ of all homogeneous quadratic polynomials $z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n$. The space $\mathfrak{g}$ is a finite-dimensional Lie subalgebra of complex-valued observables under Poisson bracket. The Lie algebra $\mathfrak{g}$ breaks naturally into 3 pieces: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ where

$$\mathfrak{k} = \text{span of } z_j\overline{z}_k, \qquad \mathfrak{p}^+ = \text{span of } z_jz_k, \qquad \mathfrak{p}^- = \text{span of } \overline{z}_j\overline{z}_k \tag{1.12}$$

Here $\mathfrak{k}$ arises as the subspace of $\phi \in \mathfrak{g}$ which Poisson commute with F so that ϕ is a conserved quantity. Then $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are the irreducible $\mathfrak{k}$-representations in $\mathfrak{g}$ complementary to $\mathfrak{k}$.

The subspace $\mathfrak{g}_\mathbb{R} \subset \mathfrak{g}$ of real-valued observables is a Lie algebra real form of $\mathfrak{g}$. We have $\mathfrak{g}_\mathbb{R} = \mathfrak{k}_\mathbb{R} \oplus \mathfrak{p}_\mathbb{R}$ where

$$\begin{aligned} \mathfrak{k}_\mathbb{R} =& \text{span of } z_j\overline{z}_k + \overline{z}_jz_k \quad \text{and} \quad i(z_j\overline{z}_k - \overline{z}_jz_k) \\ \mathfrak{p}_\mathbb{R} =& \text{span of } z_jz_k + \overline{z}_j\overline{z}_k \quad \text{and} \quad i(z_jz_k - \overline{z}_j\overline{z}_k) \end{aligned} \tag{1.13}$$

As Lie algebras, $\mathfrak{k}_\mathbb{R} \simeq \mathfrak{u}(n)$, $\mathfrak{k} \simeq \mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{g}_\mathbb{R} \simeq \mathfrak{sp}(2n, \mathbb{R})$ and $\mathfrak{g} \simeq \mathfrak{sp}(2n, \mathbb{C})$.

The Hamiltonian flow of $\mathfrak{k}_\mathbb{R}$ on $M = \mathbb{C}^n$ is the natural linear representation of the unitary group $U(n)$. The Hamiltonian flow of $\mathfrak{g}_\mathbb{R}$ is the natural linear representation of the non-compact symplectic group $Sp(2n, \mathbb{R})$. Clearly $U(n)$ is exactly the subgroup of $Sp(2n, \mathbb{R})$ which preserves the Hamiltonian F in (1.5).

We can quantize all the observables in $\mathfrak{g}$, in a way consistent with (1.7) and $\mathcal{Q}(1) = 1$, by

$$\mathcal{Q}(z_jz_k) = z_jz_k, \qquad \mathcal{Q}(z_j\overline{z}_k) = z_j\frac{\partial}{\partial z_k} + \frac{\delta_{jk}}{2}, \qquad \mathcal{Q}(\overline{z}_j\overline{z}_k) = \frac{\partial^2}{\partial z_j \partial z_k} \tag{1.14}$$

These operators obey (1.1) and $\mathcal{Q}(\phi)^\dagger = \mathcal{Q}(\overline{\phi})$ for $\phi \in \mathfrak{g}$. Moreover this condition by itself determines the inner product $\langle f|g\rangle$ uniquely. The No-Go Theorem (see e.g., [A-M]) shows that we cannot extend the quantization to all polynomial observables.

A benefit of looking at this large Lie algebra of symmetry is that we can see another source for the $\frac{1}{2}$-shift in (1.6). Indeed, the eminently reasonable values of $\mathcal{Q}(z_jz_k)$ and $\mathcal{Q}(\overline{z}_j\overline{z}_k)$ in (1.14) imply the value of $\mathcal{Q}(z_j\overline{z}_k)$ because of the Dirac axiom (1.1). So the term involving $\frac{1}{2}$ is created exactly because $\frac{\partial}{\partial z_k}$ and z_k do not commute but instead $[\frac{\partial}{\partial z_k}, z_k] = 1$.

The most convincing way to understand the $\frac{1}{2}$-shift is to introduce half-forms. This means that we replace our Hilbert space $\mathcal{H}$ of holomorphic functions on $\mathbb{C}^n$ by a new Hilbert space

$\mathcal{H}'$ of holomorphic half-forms $s = f\sqrt{\nu}$ where $f = f(z_1, \cdots, z_n)$ is still a holomorphic function and

$$\nu = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n \tag{1.15}$$

is a holomorphic n-form. Then every holomorphic vector field η acts naturally on half-forms by the Lie derivative $\mathcal{L}_\eta$ (see §5).

On half-forms, z_j and $\overline{z}_k$ quantize into the operators

$$\mathcal{Q}'(z_j) = z_j \qquad \text{and} \qquad \mathcal{Q}'(\overline{z}_j) = \mathcal{L}_{\partial_j} \tag{1.16}$$

Let $\partial_k = \frac{\partial}{\partial z_k}$. We compute $\mathcal{L}_{\partial_k}(\sqrt{\nu}) = 0$ and $\mathcal{L}_{z_j\partial_k}(\sqrt{\nu}) = \frac{1}{2}\delta_{jk}\sqrt{\nu}$. This gives

$$\mathcal{L}_{\partial_k}(f\sqrt{\nu}) = \frac{\partial f}{\partial z_k}\sqrt{\nu} \qquad \text{and} \qquad \mathcal{L}_{z_j\partial_k}(f\sqrt{\nu}) = \left(z_j\frac{\partial f}{\partial z_k} + \frac{1}{2}\delta_{jk}f\right)\sqrt{\nu} \tag{1.17}$$

On half-forms, the observables in $\mathfrak{g}$ quantize into the operators

$$\mathcal{Q}'(z_j z_k) = z_j z_k, \qquad \mathcal{Q}'(z_j\overline{z}_k) = \mathcal{L}_{z_j\partial_k}, \qquad \mathcal{Q}'(\overline{z}_j\overline{z}_k) = \mathcal{L}_{\partial_j}\mathcal{L}_{\partial_k} \tag{1.18}$$

These operators in (1.16) and (1.18) obey (1.1) and $\mathcal{Q}'(\phi)^\dagger = \mathcal{Q}'(\overline{\phi})$ where the inner product $\langle f\sqrt{\nu}|g\sqrt{\nu}\rangle$ is again given by the RHS of (1.10). In particular we get

$$\mathcal{Q}(F) = \mathcal{L}_E \quad \text{where} E = \sum_{k=1}^{n} z_k\partial_k \tag{1.19}$$

so that $\mathcal{Q}(F)$ is the Lie derivative of the holomorphic Euler vector field on $\mathbb{C}^n$.

The operators $i\mathcal{Q}(\phi)$, $\phi \in \mathfrak{g}$, give a Lie algebra representation of $\mathfrak{g}$ on H by skew-adjoint operators. THis integrates to the unitary oscillator representation

$$Mp(2m, \mathbb{R}) \to \text{Unit}\, L^2_{hol}(\mathbb{C}^m) \tag{1.20}$$

where $Mp(2m, \mathbb{R})$ is the metaplectic group which doubly covers the symplectic group $Sp(2m, \mathbb{R})$. This representation splits into exactly two irreducible pieces.

There is one more thing we can learn from the oscillator example. This is that Kaehler polarizations can turn out to be related to cotangent bundle geometry. Indeed, we gave no geometric reason for the assignments in (1.16) and (1.18). In quantizing observables on cotangent bundles T^*Q, we have the guiding philosophy that the principal symbol of $\mathcal{Q}(\phi)$ should be ϕ if ϕ is homogeneous on the fibers of the projection $T^*Q \to Q$. On a Kaehler manifold we a priori have no notion like this.

However, if (M, ω) is Kaehler with complex structure $\mathbf{J}$, then we can ask if M is a symplectic real form of the cotangent bundle T^*Z of some complex manifold Z. An obvious choice is for Z to be $(M, \mathbf{J})$ (so Z forgets ω). Then the "good" observables on M would be those that extend to holomorphic (or maybe rational) functions on T^*Z which are homogeneous on the

fibers of $T^*Z \to Z$. The good observables correspond to bona fide symbols. See §2,3 and [B3] for a way to work this out based on the Hamiltonian F. The result of this is easy to describe directly for the oscillator.

We put $Z = \mathbb{C}^n$. Let $\zeta_1, ..., \zeta_n$ be the holomorphic momentum functions on T^*Z so that $z_1, \ldots, z_n, \zeta_1, ..., \zeta_n$ are holomorphic coordinates on T^*Z and the canonical holomorphic symplectic form on T^*Z is $\Omega = \sum_{k=1}^{n} \mathrm{d}\zeta_k \wedge \mathrm{d}z_k$. Then Ω defines a Poisson bracket $\{\Phi, \Psi\}_\Omega$ on the algebra of holomorphic functions on T^*Z. We have $\{z_j, z_k\}_\Omega = \{\zeta_j, \zeta_k\}_\Omega = 0$ and $\{\zeta_j, z_k\}_\Omega = \delta_{jk}$.

We have an obvious complex Poisson algebra isomorphism

$$\alpha : \mathbb{C}[z_1, \ldots, z_n, \overline{z}_1, ..., \overline{z}_n] \to \mathbb{C}[z_1, \ldots, z_n, \zeta_1, ..., \zeta_n] \tag{1.21}$$

where $\alpha(z_k) = z_k$ and $\alpha(\overline{z}_k) = i\zeta_k$. Then $\alpha(\phi)$ is the unique extension of ϕ to a holomorphic function Φ on T^*Z with respect to the symplectic embedding b of $M = \mathbb{C}^n$ into $T^*Z = \mathbb{C}^{2n}$ where $b(w) = (w, \overline{w})$. Then

$$\alpha(z_j z_k) = z_j z_k, \qquad \alpha(z_j \overline{z}_k) = i z_j \zeta_k, \qquad \alpha(\overline{z}_j \overline{z}_k) = -\zeta_j \zeta_k \tag{1.22}$$

Now the formulas in (1.16) and (1.18) make sense as $i\zeta_k$ is the symbol of $\frac{\partial}{\partial z_k}$.

The quantization of the oscillator has manifold applications in physics – in quantum mechanics, quantum field theory, supersymmetry, etc. It also of course occupies a central place in mathematics.

III. Quantization of Hamiltonian Symmetry. To formulate a mathematical quantization problem generalizing the oscillator case, we suppress (for the time being) the Hamiltonian F and focus instead on the large finite-dimensional symmetry algebra $\mathfrak{g}$. This brings us to the notion of Hamiltonian symmetry.

Suppose we have an action of a connected Lie group G on a symplectic manifold (M, ω). We regard M as a phase space. Assume the action is *symplectic*, i.e., G preserves ω. Let $\mathfrak{g}$ be the Lie algebra of G . For each $x \in \mathfrak{g}$, we have the 1-psg (1-parameter subgroup) $\gamma_x : \mathbb{R} \to G$, $\gamma_x(t) = \exp(tx)$, generated by x. By Noether's Theorem, there is a smooth function μ^x (defined at least locally about every point of M), unique up addition of a constant, such that the Hamiltonian flow of μ^x is the action of γ_x. Then μ^x is conserved under the action of γ_x. If μ^x exists globally on M, then μ^x is called a *first integral* or *momentum function* for γ_x.

The symplectic G-action is called *Hamiltonian* if there exists a map

$$\mu^* : \mathfrak{g} \to C^\infty(M) \tag{1.23}$$

$x \mapsto \mu^x$, such that μ^x is a first integral for γ_x and $\{\mu^x, \mu^y\} = \mu^{[x,y]}$ for all $x, y \in \mathfrak{g}$, i.e., μ^* is a Lie algebra homomorphism. Then the functions μ^x define a *moment map*

$$\mu : M \to \mathfrak{g}^* \tag{1.24}$$

by $\mu^x(m) = \langle \mu(m), x \rangle$. If $\mathfrak{g}$ is semisimple then we often identify $\mathfrak{g}$ with its dual by means of the Killing form so that moment maps take values in $\mathfrak{g}$.

The moment map μ obtained in this way is G-equivariant and Poisson. Consequently the image of μ in $\mathfrak{g}^*$ is a union of coadjoint orbits. The image of the moment map is an important invariant of the action. It is easy to prove that μ is a covering onto a single coadjoint orbit if and only if the Hamiltonian action of G on M is transitive; then μ is symplectic. Such an action is called *elementary*.

Thus, symplectically and equivariantly, the elementary Hamiltonian G-spaces are, up to covering, just the coadjoint orbits of G.

Going back to our oscillator phase space, we see that the action of $Sp(2n, \mathbb{R})$ on our manifold $M = T^*\mathbb{R}^n = \mathbb{C}^n$, with the origin of $\mathbb{C}^n$ deleted, is an elementary Hamiltonian action. The moment map $\mathbb{C}^n - \{0\} \to \mathfrak{sp}(2n, \mathbb{R})$ is a 2-fold covering on the smallest (non-zero) adjoint orbit $O_\mathbb{R}$ of $Sp(2n, \mathbb{R})$. This orbit $O_\mathbb{R}$ is stable under scaling and so consists of nilpotent elements.

The quantization problem on the Hamiltonian G-space (M, ω) is to quantize the momentum functions μ^x into operators in a manner agreeable with Dirac's axioms. It is natural to study the elementary case first, as here the symmetry is largest. Thus one seeks a quantization of the functions μ^x, $x \in \mathfrak{g}$, for coadjoint orbits and their covers.

In analogy with the oscillator, we consider the case where the symmetry group G is a real semisimple Lie group $G_\mathbb{R}$ (with finite center) and M is an adjoint orbit $O_\mathbb{R}$ stable under scaling. Then $O_\mathbb{R}$ is a "nilpotent orbit" of $G_\mathbb{R}$ – see §2.

Quantization of coadjoint orbits has traditionally been considered as part of the Orbit Method in representation theory. In the Orbit method, one uses polarizations invariant under the whole symmetry group and obtains unitary representations by induction. The theory incorporates metaplectic covers and the Mackey machine. Much more can be said about the Orbit Method. We note that unitary representations attached to nilpotent orbits are called *unipotent* in representation theory.

On the other hand, coming into this problem from geometry, we have found different methods which apply (at least) to nilpotent orbits. The main idea is to transform the quantization problem on $O_\mathbb{R}$ into a quantization problem on a cotangent bundle, and then solve that problem.

IV. Outline of this Paper.

In this paper, we quantize the nilpotent orbit $O_\mathbb{R}$ of $G_\mathbb{R}$ in the case where $O_\mathbb{R}$ is strongly minimal (see §3). The oscillator phase space is the double cover of the strongly minimal nilpotent orbit of $G_\mathbb{R} = Sp(2n, \mathbb{R})$.

We assume that $\mathfrak{g}_\mathbb{R}$ is simple, the maximal compact subgroup $K_\mathbb{R}$ of $G_\mathbb{R}$ has finite center, and $G_\mathbb{R}$ is simply-connected. (Thus we exclude the oscillator case as there $K_\mathbb{R} = U(n)$.)

We obtain the analogs of the Fock space model of the quantum mechanical oscillator. We find analogs of all the features of the oscillator quantization described above in II. This is worked out in detail in this paper, with the exception of the integral formula (1.10) for the inner product which will be written up elsewhere. We work from scratch and assume no prior knowledge on existence of unitary representations.

This completes the work from [B-K4]. In [B-K4] we worked out with Kostant the results covered in §4-7 of this paper for the three cases where $G_\mathbb{R}$ is a a split group of type E_6, E_7, E_8.

We start from the fact, a product of the work of Kronheimer ([Kr]) and Vergne ([Ve]), that $O_{\mathbb{R}}$ admits a $K_{\mathbb{R}}$-invariant complex structure $\mathbf{J}$ which together with the KKS symplectic form σ gives a (positive) Kaehler structure on $O_{\mathbb{R}}$. The Vergne diffeomorphism $\mathcal{V}: O_{\mathbb{R}} \to Y$ identifies the complex manifold $(O_{\mathbb{R}}, \mathbf{J})$ with a complex homogeneous space Y of the complexification K of $K_{\mathbb{R}}$. This is a general theory that applies to every nilpotent orbit for $G_{\mathbb{R}}$ semisimple. For the oscillator, this recovers the $U(n)$-invariant Kaehler structure and the identification $T^*\mathbb{R}^n = \mathbb{C}^n$ used in II.

We outline this theory in §2 and we explain how it gives rise to an embedding of $O_{\mathbb{R}}$ into T^*Y as a totally real symplectic submanifold ([B1]). This enables us to transform the quantization problem on $O_{\mathbb{R}}$ into a quantization problem on T^*Y, as long as the Hamiltonian functions ϕ^w, $w \in \mathfrak{g}_{\mathbb{R}}$ extend from $O_{\mathbb{R}}$ to T^*Y.

An important aspect is that the Kaehler structure on $O_{\mathbb{R}}$ possesses a global Kaehler potential ρ which we argue plays the role of the Hamiltonian F. The Hamiltonian flow of ρ is the action of the center of $K_{\mathbb{R}}$ in the oscillator case. In our cases, the Hamiltonian flow of ρ lies outside the $G_{\mathbb{R}}$-action.

In §3, we specialize to the case where $O_{\mathbb{R}}$ is strongly minimal and $K_{\mathbb{R}}$ has finite center. We explain how to convert the Hamiltonian functions ϕ^w, $w \in \mathfrak{g}_{\mathbb{R}}$, on $O_{\mathbb{R}}$ into rational meromorphic functions Φ^w on the cotangent bundle of Y. We interpret the Φ^w as "pseudo-differential symbols".

To describe the symbols, we consider the Cartan decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$ (cf. (1.13)). For $x \in \mathfrak{k}_{\mathbb{R}}$, Φ^x is just the usual symbol of the holomorphic vector field η^x on Y defined by differentiating the K-action. But for $v \in \mathfrak{p}_{\mathbb{R}}$, Φ^v is a sum of two terms, each homogeneous under the fiberwise scaling action of $\mathbb{C}^*$ on the leaves of the cotangent polarization of T^*Y. The passage from the observable function ϕ^w to the symbol Φ^w preserves Poisson brackets.

The middle part §4-§7 of the paper is devoted to quantizing the symbols Φ^w, $w \in \mathfrak{g}_{\mathbb{R}}$, into skew-adjoint operators on a holomorphic half-form line bundle $\mathbf{N}^{\frac{1}{2}}$ over Y. In §5, we construct all such bundles. We find the space H of global algebraic holomorphic sections of $\mathbf{N}^{\frac{1}{2}}$ is a multiplicity free ladder representation of K. We get a simple geometric description of the sections which are the highest weight vectors.

In §4, we set up the Jordan structure that is used throughout the paper (explicity in §5 and §7). A main point is that the polynomial function P constructed in §3 is realized in terms of Jordan norms.

We construct, in Corollary 6.2 and Theorem 6.3 the pseudo-differential operators $\mathcal{Q}(\Phi^w)$ on half-forms which quantize the symbols Φ^w, or equivalently, the functions ϕ^w. Theorem 6.3 says that these operators satisfy (1.1), i.e., the operators $\pi^w = i\mathcal{Q}(\Phi^w)$ give a representation of $\mathfrak{g}$. In Theorem 6.6 we construct the $\mathfrak{g}_{\mathbb{R}}$-invariant inner product B on H. In Theorem 6.8 we compute B by giving the analog (6.30) of (1.9).

Our operators are pseudo-differential (not purely differential) in that they involve inverting the positive-spectrum "energy" operator E' which is the quantization of ρ. In fact, instead of the order two operators $\mathcal{L}_{\partial_j}\mathcal{L}_{\partial_k}$ from (1.18) we obtain order 4 differential operators divided by $E'(E'+1)$; these are "formally" of order 2. The action of the maximal compact group $K_{\mathbb{R}}$ on H is just the natural one defined by the action of $K_{\mathbb{R}}$ on Y and $\mathbf{N}^{\frac{1}{2}}$.

Theorem 6.4 says that our representation π of $\mathfrak{g}$ on H is irreducible. Also we describe the

algebra generated by the operators π^w on H. It follows in Theorem 6.6 that π integrates to give an irreducible minimal unitary representation of $G_{\mathbb{R}}$ on the Hilbert space completion $\mathcal{H}$ of H.

Next §7 is devoted to proving the results of §6. We show that our pseudo-differential operators satisfy the bracket relations of $\mathfrak{g}_{\mathbb{R}}$ by reformulating the problem and applying the generalized Capelli Identity of Kostant and Sahi ([K-S]). An important aspect of their work is that Jordan algebras provide a natural setting for generalizing the classical Capelli identity involving square matrices. The complex Jordan algebra $\mathfrak{k}_{-1}$ occurring here is semisimple (while in [B-K4] it was simple). It turns out that the simple components of $\mathfrak{k}_{-1}$ become coupled together in our calculations in a subtle way reflected by Proposition 7.8.

In §8, we compute the reproducing kernel $\mathcal{K}$ of the Hilbert space completion $\mathcal{H}$ of H. We find that $\mathcal{K}$ is a holomorphic function on $Y \times \overline{Y}$ and hence $\mathcal{H}$ consists entirely of holomorphic sections of $\mathbf{N}^{\frac{1}{2}}$. Finally, in §9 we give some examples.

Different models, or proofs of existence, for most of the unitary representations we construct have been obtained by other authors. These include Binegar, Gross, Howe, Kazhdan, Kostant, Li, Oersted, Rawnsley, Savin, Sijacki, Sternberg, Sabourin, Torasso, Vogan, Wallach, Wolf, and Zierau. Moreover in [T], Torasso constructs in a uniform manner by the Orbit Method Schroedinger type models of all minimal unitary representations. Precisely, Torasso constructs unitary irreducible representations attached to all minimal admissible nilpotent orbits of simple groups of relative rank at least three over a local field of zero characteristic. It would be very interesting to construct intertwining operators between our models.

There is a rich literature on geometric models of unitary highest weight representations, and there are many interesting ties here with our work.

This paper builds on several years of joint work with Bert Kostant on the algebraic holomorphic symplectic geometry of nilpotent orbits of a complex semisimple Lie group. This work includes [B-K1-5]. In addition §4 of this paper is joint work.

I thank Alex Astashkevich, Olivier Biquard, Murat Gunaydin, Bert Kostant, Michele Vergne, and Francois Ziegler for useful conversations relating to this work. Parts of this work were carried out during visits to Harvard (1993-94, summers of 1995 and 1996), the Institute for Advanced Study (Spring 1995) and Brown University (summer of 1997). I thank all these departments for their hospitality. I thank Mark Gotay for putting together this volume and for his comments on my paper.

I am delighted to dedicate this paper to Victor Guillemin and to be able to contribute it to this volume in his honor. In my graduate student days at MIT I was ensconced in algebraic geometry and algebraic group actions. I was symplectically agnostic. But since my symplectic conversion in the end of the last decade, I have had the opportunity to talk to Victor a lot and learn from him and his many books and papers. I thank him for warmly welcoming me as a visitor into his symplectic group.

§2. The Quantization Problem for Real Nilpotent Orbits.

The phase spaces we wish to quantize are the so-called "nilpotent orbits" of $G_{\mathbb{R}}$ where $G_{\mathbb{R}}$

is a connected non-compact real semisimple Lie group with finite center. Then $G_{\mathbb{R}}$ is a finite cover of the adjoint group of its Lie algebra $\mathfrak{g}_{\mathbb{R}}$, and $\mathfrak{g}_{\mathbb{R}}$ is semisimple. To define the nilpotent orbits we consider the coadjoint action of $G_{\mathbb{R}}$ on the dual $\mathfrak{g}_{\mathbb{R}}^*$ of $\mathfrak{g}_{\mathbb{R}}$.

Each coadjoint orbit $O_{\mathbb{R}}$ carries a natural $G_{\mathbb{R}}$-invariant symplectic form σ, often called the KKS or Lie-Poisson form. The form σ is uniquely characterized by the following property: let

$$\phi : \mathfrak{g}_{\mathbb{R}} \to C^\infty(O_{\mathbb{R}}), \qquad w \mapsto \phi^w \tag{2.1}$$

be the pullback map on functions defined by the embedding $O_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}^*$. Then ϕ is a Lie algebra homomorphism with respect to Poisson bracket on $C^\infty(O_{\mathbb{R}})$ defined by σ.

In analogy with the cotangent bundle, we wish to single out those coadjoint orbits which are *conical* in the sense that they are stable under the Euler scaling action of $\mathbb{R}^+$ (positive reals). There is a nice Lie theoretic characterization of these orbits. To get this, we first use the Killing form $(\, , \,)_{\mathfrak{g}_{\mathbb{R}}}$ to identify $\mathfrak{g}_{\mathbb{R}}^*$ with $\mathfrak{g}_{\mathbb{R}}$; we do this throughout the paper routinely. Then (conical) coadjoint orbits get identified with (conical) adjoint orbits.

An adjoint orbit is conical if and only if it consists of nilpotent elements in $\mathfrak{g}_{\mathbb{R}}$. Such orbits are called "nilpotent orbits". It is well-known in Lie theory that there are only finitely many nilpotent orbits in $\mathfrak{g}_{\mathbb{R}}$.

From now on, we take $O_{\mathbb{R}}$ to be a nilpotent orbit in $\mathfrak{g}_{\mathbb{R}}$. The quantization problem on $O_{\mathbb{R}}$ is to quantize into operators the functions ϕ^w, $w \in \mathfrak{g}_{\mathbb{R}}$. This is a reasonable goal. Ideally quantization would convert all smooth functions on $O_{\mathbb{R}}$ into operators in a manner satisfying Dirac's axioms. See, e.g., [Ki,§2.1] for a complete axiom list. But full quantization is impossible even for polynomial functions on $\mathbb{R}^2$ (the infamous No-Go Theorem). We are left hoping that, except for anomalies, finite-dimensional Hamiltonian symmetry will quantize.

In analogy with the Fock space quantization of the oscillator, we look for a Kaehler polarization of our phase space $(O_{\mathbb{R}}, \sigma)$ which is invariant under a fixed maximal compact subgroup $K_{\mathbb{R}}$ of $G_{\mathbb{R}}$. This means that we look for a $K_{\mathbb{R}}$-invariant integrable complex structure $\mathbf{J}$ on $O_{\mathbb{R}}$ such that $\mathbf{J}$ and σ together give a (positive) Kaehler structure on $O_{\mathbb{R}}$.

Fortunately, such a complex structure $\mathbf{J}$ on $O_{\mathbb{R}}$ arises from the works of Kronheimer ([Kr]) and Vergne ([Ve]) on instantons and nilpotent orbits. This gives the $K_{\mathbb{R}}$-invariant *instanton Kaehler structure* $(\mathbf{J}, \sigma)$ on $O_{\mathbb{R}}$. This structure is discussed and studied in detail in [B1]. We recall two main points.

The first point is the Vergne diffeomorphism ([Ve]). To set this up, we introduce the Cartan decomposition

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}} \tag{2.2}$$

where $\mathfrak{k}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ is the Lie algebra of $K_{\mathbb{R}}$ and $\mathfrak{p}_{\mathbb{R}}$ is its orthogonal complement with respect to the Killing form. The natural action of $K_{\mathbb{R}}$ on $\mathfrak{p}_{\mathbb{R}}$ complexifies to a complex algebraic action of K on $\mathfrak{p}$ where K is the complexification of $K_{\mathbb{R}}$ (so that K is a complex reductive algebraic group) and $\mathfrak{p} = \mathfrak{p}_{\mathbb{R}} \oplus i\mathfrak{p}_{\mathbb{R}}$.

Now the Vergne diffeomorphism

$$\mathcal{V} : O_{\mathbb{R}} \to Y \tag{2.3}$$

is a $(K_{\mathbb{R}} \times \mathbb{R}^+)$-equivariant diffeomorphism of real manifolds which maps $O_{\mathbb{R}}$ onto a K-orbit Y in $\mathfrak{p}$. Y, being a K-orbit, is manifestly a complex submanifold of $\mathfrak{p}$. Moreover $\mathbf{J}$ is the pullback through $\mathcal{V}$ of the complex structure on Y.

An important feature is that Y is stable under the Euler scaling action of $\mathbb{C}^*$ on $\mathfrak{p}$. This follows since $O_{\mathbb{R}}$ is $\mathbb{R}^+$-stable and $\mathcal{V}$ is $\mathbb{R}^+$-equivariant. Let E be the infinitesimal generator of the Euler $\mathbb{C}^*$-action so that E is the algebraic holomorphic Euler vector field on Y.

In general, the target Y of the Vergne diffeomorphism is known (by the Kostant-Sekiguchi correspondence [Sek]) but not the actual map giving $\mathcal{V}$. A little insight into $\mathcal{V}$ comes from Lie theory.

To explain this, we introduce the complexified Lie algebra $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \oplus i\mathfrak{g}_{\mathbb{R}}$. $\mathfrak{g}$ is a complex semisimple Lie algebra and carries the complex conjugation map $x + iy \mapsto \overline{x + iy} = x - iy$.

An *S-triple* in $\mathfrak{g}$ is a basis (e, h, f) of a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ which satisfies the bracket relations $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. The S-triple is *adapted* to $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{k}_{\mathbb{R}})$ if e and f are complex conjugates and $h \in i\mathfrak{k}_{\mathbb{R}}$. Given $O_{\mathbb{R}}$, we can find an S-triple $(e, h, \overline{e})$ adapted to $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{k}_{\mathbb{R}})$ such that $e + ih + \overline{e}$ lies in $O_{\mathbb{R}}$. Then Vergne's construction gives

$$\mathcal{V}(e + ih + \overline{e}) = e$$

The second point is that the Kaehler structure $(\mathbf{J}, \sigma)$ on $O_{\mathbb{R}}$ admits a global Kaehler potential ρ. This means that ρ is a smooth real valued function on $O_{\mathbb{R}}$ such that $i\partial\overline{\partial}\rho = \omega$. Moreover ρ is uniquely determined by the added condition that ρ transforms homogeneously under the Euler $\mathbb{R}^+$-action on $O_{\mathbb{R}}$. Then ρ is $K_{\mathbb{R}}$-invariant and Euler homogeneous of degree 1.

Next we examine how to use this Kaehler structure in quantization. The Vergne diffeomorphism identifies Y as $O_{\mathbb{R}}$ equipped with a complex polarization. The philosophy of Geometric Quantization now predicts that we can quantize suitably nice real-valued functions ϕ on $O_{\mathbb{R}}$ into self-adjoint operators on a Hilbert space consisting of holomorphic sections of a suitable holomorphic vector bundle over Y.

Our quantization program for $O_{\mathbb{R}}$ becomes: "quantize" each function ϕ^w, $w \in \mathfrak{g}_{\mathbb{R}}$, into a self-adjoint operator $\mathcal{Q}(\phi^w)$ on a Hilbert space $\mathcal{H}$ of square integrable holomorphic sections of a holomorphic half-form complex line bundle $\mathbf{N}^{\frac{1}{2}}$ over Y in such a way that the Dirac axiom

$$\mathcal{Q}(\phi^{[w,w']}) = i[\mathcal{Q}(\phi^w), \mathcal{Q}(\phi^{w'})] \tag{2.4}$$

is satisfied. In the course of doing this, we will end up quantizing one additional function on $O_{\mathbb{R}}$.

There are additional axioms which should also be satisfied, but these are somewhat hidden as we are only dealing with the functions ϕ^w. E.g., the axiom that the constant function 1 quantizes to the identity operator is "hidden". These "hidden axioms" are basically incorporated by our methodology developed below using symbols.

If the Hamiltonian flow of ϕ preserves $\mathbf{J}$ and ϕ is homogeneous of degree 1, then we mandate that the quantized operator is simply

$$\mathcal{Q}(\phi) = -i\mathcal{L}_{\widehat{\xi_\phi}} \tag{2.5}$$

Here $\widehat{\xi_\phi}$ is the $\mathbf{J}$-*Hamiltonian vector field* on Y defined by the condition that $\widehat{\xi_\phi}$ is holomorphic and coincides with ξ_ϕ on holomorphic functions. We write $\mathcal{L}_\eta$ for the Lie derivative operator (acting on holomorphic half-forms) with respect to a holomorphic vector field η.

Differentiating the K-action on Y we get an infinitesimal holomorphic vector field action

$$\mathfrak{k} \to \mathfrak{Vect}^{\mathrm{hol}}\, Y, \qquad x \mapsto \eta^x \tag{2.6}$$

Then $\eta^x = \widehat{\xi_{\phi^x}}$ for $x \in \mathfrak{k}_\mathbb{R}$ and so

$$\mathcal{Q}(\phi^x) = -i\mathcal{L}_{\eta^x}, \qquad \text{for } x \in \mathfrak{k}_\mathbb{R} \tag{2.7}$$

The problem, since our polarization $\mathbf{J}$ is only $K_\mathbb{R}$-invariant, is to quantize the remaining functions ϕ^v, $v \in \mathfrak{p}_\mathbb{R}$ corresponding to the second piece in the Cartan decomposition (2.2).

A key aspect of our program for quantization of real nilpotent orbits (see [B1-3]) is that we regard ρ as the Hamiltonian function on $O_\mathbb{R}$. This generalizes the case of the harmonic oscillator discussed in §1 where the Hamiltonian is the total energy. It may seem strange that the oscillator energy Hamiltonian is homogeneous quadratic while our function ρ is homogeneous linear. However the oscillator phase space $\mathbb{R}^{2n}-\{0\}$ arises as the *double* cover of a real nilpotent orbit. In that case, our linear potential function ρ does indeed pull back to a quadratic function on $\mathbb{R}^{2n}-\{0\}$, and it is easy to check that we recover the classical energy $p_1^2+q_1^2+\cdots+p_n^2+q_n^2$ (see [B3]).

In physical terms, the Hamiltonian governs the time evolution of the classical system. The quantum mechanical problem is to find the eigenvalues and eigenstates of the operator quantizing the Hamiltonian.

Thus we now demand that quantization should not only promote the symmetry functions ϕ^w to operators, but should also promote ρ to an operator. In fact the Hamiltonian flow of ρ preserves $\mathbf{J}$ and is periodic; we call this the KV (Kronheimer-Vergne) S^1-action on $O_\mathbb{R}$ ([B1]). Under $\mathcal{V}$, the KV S^1-action corresponds to the circle part of the Euler $\mathbb{C}^*$-action on Y. It follows that the $\mathbf{J}$-Hamiltonian vector field of ρ is iE. Hence

$$\mathcal{Q}(\rho) = -i\mathcal{L}_{iE} = \mathcal{L}_E \tag{2.8}$$

Let Ω be the canonical holomorphic symplectic form on T^*Y. Then Ω defines a Poisson bracket on the algebra of holomorphic functions on T^*Y, and also on the field of meromorphic functions.

A main result of [B1] is to realize the holomorphic cotangent bundle (T^*Y, Ω) as a symplectic complexification of $O_\mathbb{R}$. To do this, we push forward ρ to a smooth function ρ_Y on Y so that $\rho = \rho_Y \circ \mathcal{V}$. Next we construct the following real 1-form β on Y

$$\beta = -\frac{i}{2}(\partial - \overline{\partial})\rho_Y \tag{2.9}$$

Then β defines a smooth section of the cotangent bundle $T^*Y \to Y$.

Theorem 2.1[B1]. *The composition*

$$b : O_{\mathbb{R}} \xrightarrow{\mathcal{V}} Y \xrightarrow{\beta} T^*Y \tag{2.10}$$

embeds $O_{\mathbb{R}}$ *as a totally real symplectic submanifold of* T^*Y*. In particular,* $b^*(\mathrm{Re}\,\Omega) = \sigma$ *and* $b^*(\mathrm{Im}\,\Omega) = 0$*.*

Now, given a function ϕ on $O_{\mathbb{R}}$ which we wish to quantize, we can ask if ϕ extends to a holomorphic function Φ on T^*Y. (Such an extension, if it exists, is necessarily unique.) If so, then Φ is our candidate for the symbol of $\mathcal{Q}(\phi)$.

This philosophy is consistent with what we already found in (2.7) and (2.8). Indeed we can define the holomorphic symbols, where $x \in \mathfrak{k}$,

$$\Phi^x = \text{symbol } \eta^x \qquad \text{and} \qquad \lambda = \text{symbol } E \tag{2.11}$$

Our convention for symbols is specified by the following formula in holomorphic Darboux coordinates:

$$\text{symbol} \quad f(z_0, \dots, z_m) \frac{\partial^{k_0+\cdots+k_m}}{\partial z_0^{k_0} \cdots \partial z_m^{k_m}} = f(z_0, \dots, z_m) i^{k_0+\cdots+k_m} \zeta_0^{k_0} \cdots \zeta_m^{k_m}$$

It is easy to check ([B1,3])

Corollary 2.2.

(i) *The Kaehler potential* ρ *on* $O_{\mathbb{R}}$ *extends uniquely to a holomorphic function on* T^*Y*. Precisely,* ρ *extends to* $\lambda = \text{symbol } \mathcal{Q}(\rho)$*.*

(ii) *For* $x \in \mathfrak{k}_{\mathbb{R}}$*,* ϕ^x *extends uniquely to a holomorphic function on* T^*Y*. Precisely,* ϕ^x *extends to* $\Phi^x = \text{symbol } \mathcal{Q}(\phi^x)$*.*

In effect, β was engineered to make (i) true.

The passage from functions on $O_{\mathbb{R}}$ to holomorphic functions on T^*Y preserves Poisson brackets. I.e., if Φ_1 and Φ_2 are respectively the holomorphic extensions of two real functions ϕ_1 and ϕ_2 on $O_{\mathbb{R}}$ then $\{\Phi_1, \Phi_2\}_\Omega$ is the holomorphic extension of $\{\phi_1, \phi_2\}_\sigma$, where the subscripts indicate the symplectic form defining the Poisson brackets. This follows easily from Theorem 2.1.

Thus if all the Hamiltonian functions ϕ^w, $w \in \mathfrak{g}_{\mathbb{R}}$, extend holomorphically from $O_{\mathbb{R}}$ to T^*Y, then this in effect converts our quantization problem on $O_{\mathbb{R}}$ into a holomorphic quantization problem on T^*Y.

This brings us to the question as to whether the Hamiltonian functions ϕ^v, $v \in \mathfrak{p}_{\mathbb{R}}$, extend to holomorphic functions on T^*Y. The general answer is no. However, the better question is whether the ϕ^v extend to *meromorphic* functions Φ^v on T^*Y. In [B3], we show that the answer is yes in every case, at least if we allow the Φ^v to lie in a finite extension of the field of meromorphic functions on T^*Y. This relies on the powerful result of Biquard [Bi1, Bi2] that the homogeneous hyperkaehler potential on a complex nilpotent orbit is always a positive Nash function.

In fact, the symbols that arise here are all rational functions on T^*Y (or at least regular functions on an étale cover of a Zariski open set of T^*Y) in the sense of algebraic geometry. The holomorphic symplectic form Ω is manifestly algebraic and so Ω defines a Poisson bracket on the algebra $R(T^*Y)$ of algebraic holomorphic functions on T^*Y and also on the field $\mathbb{C}(T^*Y)$ of rational functions on T^*Y.

In the next section, we explain in detail how this works for the smallest orbits.

§3. Pseudo-Differential Symbol Realization of $\mathfrak{g}_{\mathbb{R}}$ for $O_{\mathbb{R}}$ Strongly Minimal.

Each nilpotent orbit $O_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ lies in a unique complex adjoint orbit $O \subset \mathfrak{g}$. Then O is a complex nilpotent orbit (i.e., O consists of nilpotent elements in $\mathfrak{g}$). We call O the *complexification* of $O_{\mathbb{R}}$. The nilpotent elements in $\mathfrak{g}$ are characterized by the property that their adjoint orbits are stable under the scaling action of $\mathbb{C}^*$.

We assume from now on that the complex Lie algebra $\mathfrak{g}$ is simple. Let G be the adjoint group of $\mathfrak{g}$. Then G is a connected complex semisimple algebraic group with Lie algebra $\mathfrak{g}$ and complex conjugation on $\mathfrak{g}$ defines a complex conjugation map $g \mapsto \overline{g}$ on G. Let $(\ ,\ )_{\mathfrak{g}}$ be the complex Killing form of $\mathfrak{g}$. We often identify $\mathfrak{g}$ with $\mathfrak{g}^*$ and $\mathfrak{k}$ with $\mathfrak{k}^*$ by means of $(\ ,\ )_{\mathfrak{g}}$.

Recall from §2 that $K_{\mathbb{R}} \subset G_{\mathbb{R}}$ is a fixed maximal compact subgroup with complexification K. We have natural maps $G_{\mathbb{R}} \to G$ and $K \to G$ and both maps have finite kernel.

Since $\mathfrak{g}$ is a simple Lie algebra, the adjoint representation of G on $\mathfrak{g}$ is irreducible. The orbit $O_{\min}$ of highest weight vectors is then nilpotent, as it is the orbit of a highest root vector. Moreover, $O_{\min}$ is minimal among all non-zero nilpotent orbits in the sense that it lies in the closure of every non-zero nilpotent orbit. It follows that $O_{\min}$ is the *unique* (non-zero) minimal nilpotent orbit.

We will call a real nilpotent orbit $O_{\mathbb{R}}$ *strongly minimal* if the complexification of $O_{\mathbb{R}}$ is $O_{\min}$. In Theorem 4.1 below we recall from [B-K5] the classification of strongly minimal real nilpotent orbits. For $O_{\mathbb{R}}$ strongly minimal, formulas for $\mathcal{V}$ and ρ are easy to write down because the action of $K_{\mathbb{R}} \times \mathbb{R}^+$ is transitive on $O_{\mathbb{R}}$. (However, for general $O_{\mathbb{R}}$ the action is not transitive, and working out $\mathcal{V}$ and ρ is a hard open problem.)

As $\mathfrak{g}$ is simple, there are just two possibilities for the center of $K_{\mathbb{R}}$: either (i) $\operatorname{Cent} K_{\mathbb{R}}$ is a circle subgroup or (ii) $\operatorname{Cent} K_{\mathbb{R}}$ is finite. These cases correspond exactly to the nature of the irreducible symmetric space $G_{\mathbb{R}}/K_{\mathbb{R}}$, so that $G_{\mathbb{R}}/K_{\mathbb{R}}$ is Hermitian in (i) and non-Hermitian in (ii). Accordingly, we call the complex symmetric pair $(\mathfrak{g}, \mathfrak{k})$ Hermitian or non-Hermitian.

For each $v \in \mathfrak{p}$, let f_v be the linear function on $\mathfrak{p}$ defined by $f_v(u) = (v, u)_{\mathfrak{g}}$. Then by restriction to Y we get a K-equivariant complex linear map

$$\mathfrak{p} \to R(Y), \qquad v \mapsto f_v \tag{3.1}$$

Every algebraic holomorphic function on Y defines an algebraic holomorphic function on T^*Y by pullback through the projection $T^*Y \to Y$.

From now on in §3, we assume that $O_{\mathbb{R}}$ is strongly minimal and the center of $K_{\mathbb{R}}$ is finite.

Theorem 3.1. *Let $v \in \mathfrak{p}_{\mathbb{R}}$ and $x \in \mathfrak{k}_{\mathbb{R}}$.*

(i) *Recall the embedding* $b : O_{\mathbb{R}} \to T^*Y$ *from* (2.10). *Each function* ϕ^v *on* $O_{\mathbb{R}}$ *extends uniquely to a rational function* Φ^v *on* T^*Y. *Set* $\Phi^{x+v} = \Phi^x + \Phi^v$ *where* Φ^x *was defined in* (2.11). *The resulting linear map*

$$\mathfrak{g}_{\mathbb{R}} \to \mathbb{C}(T^*Y), \qquad w \mapsto \Phi^w \tag{3.2}$$

is a 1*-to-*1 *real Lie algebra homomorphism with respect to the Poisson bracket on* $\mathbb{C}(T^*Y)$ *defined by* Ω.

(ii) *Each rational function* Φ^v *is everywhere defined on the Zariski open dense complex algebraic submanifold*

$$M = \{m \in T^*Y \mid \lambda(m) \neq 0\} \tag{3.3}$$

so that Φ^v *is algebraic holomorphic on* M.

(iii) *We have*

$$\Phi^v = f_v + g_v \tag{3.4}$$

where g_v *is an algebraic holomorphic function on* M *which is homogeneous of degree* 2 *with respect to the Euler* $\mathbb{C}^*$*-action on the fibers of the the projection* $M \hookrightarrow T^*Y \to Y$.

Proof. This is proven in a more general setting in [A-B1]. ■

We write $R(X)$ for the algebra of algebraic holomorphic functions on a complex algebraic variety X. Recall from (2.11) that $\lambda \in R(T^*Y)$ is the symbol of the Euler vector field.

Lemma 3.2. *We have* $R(M) = R(T^*Y)[\lambda^{-1}]$.

Proof. This follows easily since M is the complement of the irreducible divisor $(\lambda = 0)$ in the smooth (and hence normal) variety T^*Y. ■

It is natural now to extend (3.2) $\mathbb{C}$-linearly so that $\Phi^{x+iy} = \Phi^x + i\Phi^y$ for $x, y \in \mathfrak{g}_{\mathbb{R}}$. This is consistent with (2.11). We have the complexified Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{3.5}$$

Corollary 3.3. *The map* (3.2) *extends to a* 1*-to-*1 *complex Lie algebra homomorphism*

$$\mathfrak{g} \to R(T^*Y)[\lambda^{-1}], \qquad z \mapsto \Phi^z \tag{3.6}$$

Then for $v \in \mathfrak{p}$ *we have again the same formula* (3.4).

The significance of Corollary 3.3 is that we can regard functions in $R(T^*Y)[\lambda^{-1}]$ as "pseudo-differential" symbols; cf. §6.

We have now, in Theorem 3.1 and Corollary 3.3, transformed our original problem of quantizing the functions ϕ^w, $w \in \mathfrak{g}_{\mathbb{R}}$, on $O_{\mathbb{R}}$ into the problem of quantizing the rational functions Φ^w, $w \in \mathfrak{g}_{\mathbb{R}}$, on T^*Y. We mandate

$$\mathcal{Q}(\Phi^w) = \mathcal{Q}(\phi^w)$$

The new problem lies in the holomorphic symplectic category: the problem is to quantize each Φ^w into a self-adjoint operator $\mathcal{Q}(\Phi^w)$ on a Hilbert space consisting of holomorphic sections of a holomorphic half-form bundle on Y.

The advantage of the new problem is that Φ^w is already a symbol, and so we can try to quantize it by constructing reasonable quotients of differential operators with symbol Φ^w. We emphasize that (3.4) says that Φ^v, $v \in \mathfrak{p}_{\mathbb{R}}$, is *not* a principal symbol, but instead is a sum of two principal symbols f_v and g_v. We will get around this by a naive trick: we will quantize f_v and g_v separately and then add the answers.

Since f_v is just a holomorphic function on Y, we mandate that the quantization of f_v is $\mathcal{Q}(f_v) = f_v$, i.e., $\mathcal{Q}(f_v)$ is the operator defined by multiplication by f_v.

The aim of the rest of this section is to state a formula for the symbols g_v. We want to express g_v in terms of the basic symbols f_v, $v \in \mathfrak{p}$, Φ^x, $x \in \mathfrak{k}$, and λ since we already know how to quantize these symbols. To work this out, we construct a set of local (étale) coordinates on T^*Y consisting of basic symbols in Lemma 3.5 below.

We begin by setting up some of the Lie theoretic structure associated to $O_{\mathbb{R}}$ following [B-K4,§2]. We will make use of this throughout the paper. We note that the discussion of $O_{\min}$ in [B-K4,§2] was in the same generality we have here, and it was only from §3 onwards in that paper that the work specialized to the three cases where $G_{\mathbb{R}}$ is split of type E_6, E_7 or E_8.

To begin with we have

$$O_{\mathbb{R}} = O_{\min} \cap \mathfrak{g}_{\mathbb{R}} = G_{\mathbb{R}} \cdot (e + ih + \overline{e}) \qquad \text{and} \qquad Y = O_{\min} \cap \mathfrak{p} = K \cdot e \tag{3.7}$$

where $(e, h, \overline{e})$ are chosen as in §2. Then

$$\mathfrak{s} = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}\overline{e} \tag{3.8}$$

is the corresponding $\mathfrak{sl}(2, \mathbb{C})$-subalgebra. We assume from now on that $(\, ,\,)_{\mathfrak{g}}$ is rescaled so that $(e, \overline{e})_{\mathfrak{g}} = 1$.

The action of $\operatorname{ad} h$ on $\mathfrak{g}$ is diagonalizable with spectrum $\{\pm 2, \pm 1, 0\}$ so that we have the 5-grading

$$\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \tag{3.9}$$

where the subscripts indicate the corresponding eigenvalues. Then $\mathfrak{g}_s = \mathfrak{k}_s \oplus \mathfrak{p}_s$ and

$$\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_{-1} \qquad \text{and} \qquad \mathfrak{p} = \mathfrak{p}_2 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_0 \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_{-2} \tag{3.10}$$

Clearly then $\mathfrak{k}_{\pm 1}$ and $\mathfrak{p}_{\pm 1}$ are abelian Lie subalgebras. We recall from [B-K4,§2.2.-2.4]

Lemma 3.4. *The spaces $\mathfrak{g}_{\pm 2}$ are 1-dimensional with*

$$\mathfrak{g}_2 = \mathfrak{p}_2 = \mathbb{C}e \qquad \textit{and} \qquad \mathfrak{g}_{-2} = \mathfrak{p}_{-2} = \mathbb{C}\overline{e} \tag{3.11}$$

We have $\dim_{\mathbb{C}} \mathfrak{p}_s = \dim_{\mathbb{C}} \mathfrak{p}_{-s}$ and $\dim_{\mathbb{C}} \mathfrak{k}_s = \dim_{\mathbb{C}} \mathfrak{k}_{-s}$. The Lie bracket defines a perfect pairing $\mathfrak{k}_1 \times \mathfrak{p}_1 \to \mathbb{C}e$. Thus we may define

$$m = \dim_{\mathbb{C}} \mathfrak{p}_{\pm 1} = \dim_{\mathbb{C}} \mathfrak{k}_{\pm 1} \tag{3.12}$$

The subspace $\mathfrak{g}_2 \oplus \mathfrak{g}_1$ is a $(2m+1)$-dimensional Heisenberg Lie algebra with center $\mathfrak{g}_2$. We have $\mathfrak{g} = \mathfrak{g}^{\overline{e}} \oplus \mathbb{C}h \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_1$. Consequently

$$\dim_{\mathbb{C}} O_{\min} = 2m + 2 \tag{3.13}$$

In particular

$$\dim_{\mathbb{C}} \ Y = \frac{1}{2} \dim_{\mathbb{C}} O_{\min} = m + 1 \tag{3.14}$$

Now we take a basis $v_1, \ldots, v_m$ of $\mathfrak{p}_1$. We put $v_0 = e$. The corresponding regular functions on Y defined by (3.1) are

$$f_0 = f_{v_0}, \qquad f_1 = f_{v_1}, \quad \ldots, \quad f_m = f_{v_m} \tag{3.15}$$

These form a system of local coordinates on Y by ([B-K3, Prop. 5.2]). In fact we get an isomorphism of varieties

$$Y^o \to \mathbb{C}^* \times \mathbb{C}^m, \qquad y \mapsto (f_0(y), f_1(y), \ldots, f_m(y)) \tag{3.16}$$

where $Y^o \subset Y$ is the open set given by

$$Y^o = (f_0 \neq 0) \tag{3.17}$$

In our local coordinates we have

$$\eta^x = \sum_{k=0}^{m} (\eta^x f_k) \frac{\partial}{\partial f_k} \tag{3.18}$$

Let $\{x_1, \ldots, x_m\}$ be the basis of $\mathfrak{k}_1$ such that $[x_i, v_j] = \delta_{ij} e$. Then in terms of our local coordinates $f_0, f_1, \ldots, f_m$ the expressions for our vector fields E and η^{x_i}, $i = 1, \ldots, m$ are

$$E = \sum_{i=0}^{m} f_i \frac{\partial}{\partial f_i} \qquad \text{and} \qquad \eta^{x_i} = f_0 \frac{\partial}{\partial f_i} \tag{3.19}$$

It follows easily from these formulas that

Lemma 3.5. *The $2m+2$ functions $f_0, f_1, \ldots, f_m, \lambda, \Phi^{x_1}, \ldots, \Phi^{x_m}$ form a system of local étale coordinates on T^*Y.*

The single function g_{v_0} determines all the functions g_v because of the K-action; indeed, $g_{[x,v]} = \{\Phi^x, g_v\}_\Omega$ for $x \in \mathfrak{k}$. To state our formula for g_{v_0} we need one more ingredient, the polynomial function function P defined below.

The vector fields η^x, $x \in \mathfrak{k}$ define a natural complex algebra homomorphism from the universal enveloping algebra $\mathcal{U}(\mathfrak{k})$ to the algebra $\mathcal{D}(Y)$ of algebraic holomorphic differential operators on Y. So in particular we get a representation

$$\pi_K : \mathcal{U}(\mathfrak{k}) \to \mathrm{End}\ R(Y) \tag{3.20}$$

On the symbol level, (3.20) corresponds to the graded Poisson algebra homomorphism

$$\Phi_K : S(\mathfrak{k}) \to R(T^*Y) \tag{3.21}$$

defined by $\Phi_K(x) = \Phi^x$ for $x \in \mathfrak{k}$.

The adjoint action of $\mathfrak{g}$ defines a a complex algebra homomorphism $\mathrm{ad} : \mathcal{U}(\mathfrak{k}) \to \mathrm{End}\,\mathfrak{p}$, $Q \mapsto \mathrm{ad}\,Q = \mathrm{ad}_Q$. Let P be the polynomial function on $\mathfrak{k}_{-1}$ defined by

$$\frac{1}{4!}\mathrm{ad}_y^4(e) = P(y)e \tag{3.22}$$

where $y \in \mathfrak{k}_{-1}$. Then P is homogeneous of degree 4.

We have a perfect pairing

$$\mathfrak{k}_1 \times \mathfrak{k}_{-1} \to \mathbb{C} \tag{3.23}$$

defined by the Killing form $(\ ,\ )_\mathfrak{g}$ as in [B-K4,§2.5]. This gives an identification of $S(\mathfrak{k}_1)$ with the algebra of polynomial functions on $\mathfrak{k}_{-1}$. This identification places $P \in S^4(\mathfrak{k}_1)$ so that we may write

$$P = P(x_1, \ldots, x_m) \qquad \text{and} \qquad \Phi_K P = P(\Phi^{x_1}, \ldots, \Phi^{x_m}) \tag{3.24}$$

Theorem 3.6. *The function g_v in Theorem 3.1(iii) is given for $v = v_0 = e$ by*

$$g_{v_0} = -\frac{1}{\lambda^2}\frac{\Phi_K P}{f_0} \tag{3.25}$$

Proof. A more general result is proven in [A-B2]. ■

In the next section, we set up the Jordan algebra machinery which gives us a useful and computable way to understand the polynomial P. In §5, we already use this machinery to classify half-form bundles on Y. The reader eager to see how we quantize the symbols g_v and then the symbols Φ^w can skip ahead to Lemma 5.3 and Proposition 5.5 and then to §6.

§4. Complex Minimal Nilpotent Orbits and Jordan Algebras.

Our first aim in this section is to classify the real simple Lie algebras $\mathfrak{g}_{\mathbb{R}}$ which possess a strongly minimal real nilpotent orbit. This amounts to classifying $\mathfrak{g}_{\mathbb{R}}$ such that $O_{\min}$ has real points because of (3.7). This classification, recalled in Theorem 4.1 below, uses the geometry of $O_{\min}$.

Any complex nilpotent orbit O, and so in particular $O_{\min}$, is a quasi-affine smooth locally closed complex algebraic subvariety in $\mathfrak{g}$. This follows since the adjoint action of G on $\mathfrak{g}$ is complex algebraic. Furthermore, O is an algebraic holomorphic symplectic manifold with respect to its G-invariant holomorphic KKS symplectic form Σ (cf. [B-K1]). The G-action on O is Hamiltonian with holomorphic moment map given by the embedding $O \subset \mathfrak{g}$.

Let $\mu : \mathfrak{g} \to \mathfrak{k}$ be the projection defined by (3.5). Then the composite map $\mu : O_{\min} \to \mathfrak{g} \to \mathfrak{k}$ is the moment map for the Hamiltonian K-action on $O_{\min}$. Let $\mathcal{N}(\mathfrak{k})$ be the cone of nilpotent elements in $\mathfrak{k}$.

Theorem 4.1[B-K5]. *The following conditions are equivalent:*

(i) $O_{\min} \cap \mathfrak{g}_{\mathbb{R}}$ *is empty*

(ii) K *has a Zariski open orbit on* $O_{\min}$

(iii) $\mu(O_{\min}) \subset \mathcal{N}(\mathfrak{k})$

and imply that the principal isotropy group of K *on* $O_{\min}$ *is* $K^{\mathfrak{s}}$ *where* $\mathfrak{s}$ *was defined in* (3.8.

The complete list of all complex symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ *(with* $\mathfrak{g}$ *simple) which satisfy* (i)-(iii) *is:*

(a) $(\mathfrak{sl}(2n, \mathbb{C}), \mathfrak{sp}(2n, \mathbb{C}))$, *where* $n \geq 2$

(b) $(\mathfrak{so}(p+1, \mathbb{C}), \mathfrak{so}(p, \mathbb{C}))$, *where* $p \geq 3$

(c) $(\mathfrak{sp}(2p+2q, \mathbb{C}), \mathfrak{sp}(2p, \mathbb{C}) + \mathfrak{sp}(2q, \mathbb{C}))$, *where* $p, q \geq 1$

(d) $(F_4, \mathfrak{so}(9, \mathbb{C}))$

(e) (E_6, F_4)

Each pair $(\mathfrak{g}, \mathfrak{k})$ *in this list is non-Hermitian.*

From the point of view of representation theory, the condition that $O_{\min} \cap \mathfrak{g}_{\mathbb{R}}$ is non-empty is very natural. To explain this, we recall the theory of the associated variety.

Suppose $\pi_o : G_{\mathbb{R}} \to \operatorname{Unit} \mathcal{H}$ is an irreducible unitary representation. Let $H \subset \mathcal{H}$ be the space of $K_{\mathbb{R}}$-finite vectors with its natural $(\mathfrak{g}, K)$-module structure; H is then the Harish-Chandra module of the representation. Differentiation of the group representation gives a Lie algebra representation of $\mathfrak{g}_{\mathbb{R}}$ on H and so a representation $\widetilde{\pi} : \mathcal{U}(\mathfrak{g}) \to \operatorname{End} H$ of the universal enveloping algebra. The annihilator $\mathcal{I}$ is then the primitive ideal attached to π_o. The graded ideal $\operatorname{gr} \mathcal{I}$ cuts out a closed complex algebraic subvariety $\mathcal{V}(\operatorname{gr} \mathcal{I}) \subset \mathfrak{g}^* \simeq \mathfrak{g}$ called the associated variety of $\mathcal{I}$. Since π_o admits a central character, it follows that $\mathcal{V}(\operatorname{gr} \mathcal{I})$ is a union of complex nilpotent orbits. A basic result (due independently to Borho and J.L. Brylinski, to Ginzburg, and to Joseph) is that $\mathcal{V}(\operatorname{gr} \mathcal{I})$ is in fact the closure of a *single* nilpotent orbit, which is then called the associated complex nilpotent orbit of $\mathcal{H}$ and H.

The following observation is an easy consequence of the theory of associated varieties. For instance, it is a corollary of [Vo2, Theorem 8.4].

Lemma 4.2. *Suppose $G_\mathbb{R}$ admits an irreducible unitary representation with associated complex nilpotent orbit $O \subset \mathfrak{g}$. Then $O \cap \mathfrak{g}_\mathbb{R}$ is non-empty, i.e., O has a real form with respect to $\mathfrak{g}_\mathbb{R}$.*

We will call an irreducible unitary representation $\pi_o : G_\mathbb{R} \to \mathrm{Unit}\,\mathcal{H}$ *minimal* if its associated nilpotent orbit is $O_{\min}$ and also the image of $\widetilde{\pi} : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}\,H$ has no zero-divisors , i.e., the annihilator $\mathcal{I}$ of $\widetilde{\pi}$ is completely prime. For $\mathfrak{g}$ not of type A_n, π_o is minimal if and only if $\mathcal{I}$ is the Joseph ideal. So Lemma 4.2 says that a necessary (but not sufficient) geometric requirement for $G_\mathbb{R}$ to admit a minimal representation is that $O_{\min} \cap \mathfrak{g}_\mathbb{R}$ is non-empty.

From now on in this paper, we assume that $O_\mathbb{R}$ is strongly minimal and $(\mathfrak{g}, \mathfrak{k})$ is non-hermitian. For convenience, we also take $G_\mathbb{R}$ to be simply-connected. There is no problem in this as the universal cover has finite center.

We freely identify $O_\mathbb{R}$ with Y via the Vergne diffeomorphism (2.3). Using Lemma 3.4 and (3.14), we find the dimension of Y is given by:

$$\begin{array}{cccccccc} \mathfrak{g} & \mathfrak{sl}(n,\mathbb{C}) & \mathfrak{so}(n,\mathbb{C}) & \mathfrak{sp}(2n,\mathbb{C}) & G_2 & F_4 & E_6 & E_7 & E_8 \\ \dim_\mathbb{C} Y & n-1 & n-3 & n & 3 & 8 & 11 & 17 & 29 \end{array} \tag{4.1}$$

Next we want to develop the Jordan theory interpretation of the polynomial P defined in (3.22). We find in Proposition 4.4 below a Jordan structure on the space $\mathfrak{k}_{-1}$ from (3.10).

There is a natural symmetry group acting on the space $\mathfrak{k}_{-1}$, namely the isotropy group $K_0 = K^h$ for the adjoint action of K. We use this symmetry throughout the paper. K_0 is a closed reductive complex algebraic subgroup of K with Lie algebra $\mathfrak{k}_0$. Also K_0 is connected; this follows immediately from the fact that the adjoint orbit $K \cdot h$ is simply-connected. Basic constructions like (3.10), (3.16), and (3.17) break the K-symmetry but not the K_0-symmetry.

In particular, K_0 acts on $\mathfrak{p}_2$ by a (non-trivial) character

$$\chi : K_0 \to \mathbb{C}^* \tag{4.2}$$

so that $a \cdot e = \chi(a)e$ for $a \in K_0$. Let

$$K_0' = \text{kernel of } \chi = K^{\mathfrak{s}} \tag{4.3}$$

and let $\mathfrak{k}_0' = \mathfrak{k}^{\mathfrak{s}}$ be the Lie algebra of K_0'. We get an orthogonal decomposition

$$\mathfrak{k}_0 = \mathfrak{k}_0' \oplus \mathbb{C}h \tag{4.4}$$

Since $\mathfrak{k}_1$ is abelian, we have a natural identification

$$\mathcal{U}(\mathfrak{k}_1) = S(\mathfrak{k}_1) \tag{4.5}$$

So in particular, P defines an element of $\mathcal{U}(\mathfrak{k}_1)$. We recall from [B-K4,§2.2-2.6]:

Proposition 4.3. *The polynomial $P \in S^4(\mathfrak{k}_1)$ defined in (3.22) is semi-invariant under K_0 and transforms by the character χ^2. Moreover, P is, up to scaling, the unique K_0-semi-invariant polynomial in $S(\mathfrak{k}_1)$ such that*

$$\mathrm{ad}_P(\mathbb{C}\overline{e}) = \mathbb{C}e \tag{4.6}$$

We recall some work from [B-K4, §2.6-8]. We found with Kostant a nilpotent element $e_{\mathfrak{k}} \in \mathfrak{k}_1$ such that

$$\mathfrak{l} = \mathbb{C}h \oplus \mathbb{C}e_{\mathfrak{k}} \oplus \mathbb{C}\overline{e}_{\mathfrak{k}} \tag{4.7}$$

is a complex Lie subalgebra in $\mathfrak{k}$ isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ and $(2h, -e_{\mathfrak{k}}, \overline{e}_{\mathfrak{k}})$ is an S-triple basis of $\mathfrak{l}$. We normalized the choice of $e_{\mathfrak{k}}$ so that $\frac{1}{4!}(\operatorname{ad} e_{\mathfrak{k}})^4(\overline{e}) = e$ and $\frac{1}{4!}(\operatorname{ad} \overline{e}_{\mathfrak{k}})^4(e) = \overline{e}$. Hence

$$P(\overline{e}_{\mathfrak{k}}) = 1 \tag{4.8}$$

We note that the nilpotents $e \in \mathfrak{p}$ and $e_{\mathfrak{k}} \in \mathfrak{k}$ were called, respectively, z and e in [B-K4].

Then we showed that $(\mathfrak{k}, \mathfrak{k}_0)$ is a Hermitian symmetric pair of tube type with rank q where $q \leq 4$. In addition, the pair $(\mathfrak{k}_0, \mathfrak{k}^{\mathfrak{l}})$ is a complex symmetric pair so that we have a complex Cartan decomposition $\mathfrak{k}_0 = \mathfrak{k}^{\mathfrak{l}} \oplus \mathfrak{r}$ where $[\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{k}^{\mathfrak{l}}$.

Consequently, elaborating on [B-K4, Proposition 2.8] we get

Proposition 4.4. *The Tits-Kantor-Koecher construction gives $\mathfrak{k}_{-1}$ the structure of a complex semisimple Jordan algebra with $K^{\mathfrak{l}}$-invariant Jordan product defined by*

$$[x, \overline{e}_{\mathfrak{k}}] \circ [y, \overline{e}_{\mathfrak{k}}] = [x, [y, \overline{e}_{\mathfrak{k}}]] \tag{4.9}$$

where $x, y \in \mathfrak{r}$. The Jordan identity element is $\overline{e}_{\mathfrak{k}}$. The Jordan algebra degree of $\mathfrak{k}_{-1}$ is

$$\deg \mathfrak{k}_{-1} = q = \textit{rank } (\mathfrak{k}, \mathfrak{k}_0) \leq 4 \tag{4.10}$$

The T-K-K theory identifies $\mathfrak{k}_{-1}$ as the complexification $\mathcal{J}_{\mathbb{C}}$ of a real Euclidean Jordan algebra $\mathcal{J}$. The book [F-K] is an excellent reference for the theory of real Euclidean and complex semisimple Jordan algebras.

Next we write out the decomposition of $\mathfrak{k}_{-1}$ into a direct sum of complex simple Jordan subalgebras:

$$\mathfrak{k}_{-1} = \mathfrak{j}_{[1]} \oplus \cdots \oplus \mathfrak{j}_{[\ell]} \tag{4.11}$$

Then each space $\mathfrak{j}_{[n]}$ carries an irreducible representation of $\mathfrak{k}_0$. Let q_n be the degree of $\mathfrak{j}_{[n]}$; then

$$q_1 + \cdots + q_\ell = q \tag{4.12}$$

Let $P_{[n]}$ be the Jordan norm of $\mathfrak{j}_{[n]}$; then $P_{[n]}$ has degree q_n.

Proposition 4.5. *The polynomial $P \in S^4(\mathfrak{k}_1)$, constructed in* (3.22)*, as a function on $\mathfrak{k}_{-1}$, is uniquely expressible as a monomial*

$$P = P_{[1]}^{w_1} \cdots P_{[\ell]}^{w_\ell} \tag{4.13}$$

in the Jordan norms $P_{[n]}$ of the simple components $\mathfrak{j}_{[n]}$ of $\mathfrak{k}_{-1}$. Every exponent $w_1,\ldots,w_\ell$ is positive.

Proof. In the Tits-Kantor-Koecher construction, the simple Lie components of $\mathfrak{k}$ correspond to the simple Jordan components of $\mathfrak{k}_{-1}$. I.e., $\mathfrak{j}_{[n]}=\mathfrak{k}_{-1}\cap\mathfrak{r}_{[n]}$ where $\mathfrak{k}=\mathfrak{r}_{[1]}\oplus\cdots\oplus\mathfrak{r}_{[\ell]}$ is the decomposition of $\mathfrak{k}$ into complex simple Lie subalgebras. Then we get also the decomposition $\mathfrak{k}_0=\mathfrak{t}_{[1]}\oplus\cdots\oplus\mathfrak{t}_{[\ell]}$ into a direct sum of subalgebras where $\mathfrak{t}_{[n]}=\mathfrak{k}_0\cap\mathfrak{r}_{[n]}$. Each pair $(\mathfrak{r}_{[n]},\mathfrak{t}_{[n]})$ is complex symmetric of tube type. Then any polynomial function on $\mathfrak{j}_{[n]}$ semi-invariant under $\mathfrak{t}_{[n]}$ is a polynomial in $P_{[n]}$; see, e.g. [K-S, Th. 0]. It follows that P is of the form $P=cP_{[1]}^{w_1}\cdots P_{[\ell]}^{w_\ell}$ for some scalar c. But $c=1$ since $P(\overline{e}_{\mathfrak{k}})=P_{[1]}(\overline{e}_{\mathfrak{k}})=\cdots=P_{[\ell]}(\overline{e}_{\mathfrak{k}})=1$. Finally the fact that h has non-zero projection to each component $\mathfrak{r}_{[1]},\ldots,\mathfrak{r}_{[\ell]}$ implies that each $w_1,\ldots,w_\ell$ is non-zero. ■

Notice that (4.13) *defines* the exponents $w_1,\ldots,w_\ell$ and gives the numerical equality

$$q_1w_1+\cdots q_\ell w_\ell=4 \tag{4.14}$$

In [B-K4], we quantized with Kostant the real form $O_{\mathbb{R}}$ of $O_{\min}$ in the three cases where $\mathcal{J}_{\mathbb{C}}$ is a simple complex Jordan algebra of degree 4 so that $\ell=1$ and $P=P_{[1]}$. In this paper we treat the general case where P may factor non-trivially.

We proceed in the rest of this section to make an explicit list of the Jordan algebras occurring here. The associations we get between exceptional Lie algebras and Jordan algebras are in many cases already familiar from the constructions discovered by Tits, Kantor, Koecher, and Allison and Falkner to produce exceptional Lie algebras out of Jordan algebras.

In our tables we adopt the following conventions. We write $\mathfrak{so}_p$, $\mathfrak{sp}_{2p}$, $\mathfrak{sl}_p$, G_2, F_4, E_6, E_7, E_8 for the corresponding complex Lie algebras. Also $S^n_o\mathbb{C}^p$ denotes the irreducible representation of $\mathfrak{so}(p,\mathbb{C})$ satisfying $S^n_o\mathbb{C}^p+S^{n-2}\mathbb{C}^p\simeq S^n\mathbb{C}^p$ while $\wedge^n_o\mathbb{C}^{2p}$ denotes the irreducible representation of $\mathfrak{sp}(2p,\mathbb{C})$ satisfying $\wedge^n_o\mathbb{C}^{2p}+\wedge^{n-2}\mathbb{C}^{2p}\simeq\wedge^n\mathbb{C}^{2p}$ where $n\le p$.

A complete list of all non-isomorphic formally real simple Jordan algebras of degree ≤ 4 follows immediately from the Tits-Kantor-Koecher theory and the known list of all irreducible Hermitian symmetric tube domains (see [H], p. 528, Example 4 and §6.4, pp. 518-520). We give this list in Table 4.6 together with the corresponding pair $(\mathfrak{k},\mathfrak{k}_0)$. The number d arises in the following way. The restricted root system for the pair $(\mathfrak{k},\mathfrak{k}_0)$ is of type C_q where q is the degree of $\mathcal{J}$ and then the long roots have multiplicity 1 while the short roots all have common multiplicity d.

In Table 4.6, $\mathbb{R}^p$, $p\ge 1$, is the p-dimensional real Jordan algebra associated to the Euclidean norm and $\mathrm{Herm}(n,\mathbb{F})$ is the real Jordan algebra of $n\times n$ hermitian matrices over $\mathbb{F}$ where $\mathbb{H}$ and $\mathbb{O}$ denote the quaternions and the octonions (the Cayley numbers) respectively. Then $\mathrm{Herm}(3,\mathbb{O})$ is the exceptional 27-dimensional Jordan algebra while all the others in Table 4.6 are special (i.e., arise in the standard way from associative algebras). The last column in Table 4.6 gives a name to the Jordan norm of $\mathcal{J}$.

Table 4.6. All Simple Euclidean Real Jordan Algebras $\mathcal{J}$ of rank ≤ 4

$\mathcal{J}$	$\dim_{\mathbb{R}} \mathcal{J}$	d	$\mathfrak{k}$	$\mathfrak{k}_0$	$\deg \mathcal{J}$	Norm
$\mathcal{J}(1) = \mathbb{R}$	1	0	$\mathfrak{sl}_2$	$\mathfrak{so}_2$	1	P_1
$\mathcal{J}(2;p) = \mathbb{R}^{p-2}$, $p \geq 5$	$p-2$	$p-4$	$\mathfrak{so}_p$	$\mathfrak{so}_{p-2} \oplus \mathfrak{so}_2$	2	$P_{2;p}$
$\mathcal{J}(3,\mathbb{R}) = \mathrm{Herm}(3,\mathbb{R})$	$6 = 3+3d$	1	$\mathfrak{sp}_6$	$\mathfrak{gl}_3$	3	$P_{3;\mathbb{R}}$
$\mathcal{J}(3,\mathbb{C}) = \mathrm{Herm}(3,\mathbb{C})$	$9 = 3+3d$	2	$\mathfrak{sl}_6$	$\mathfrak{s}(\mathfrak{gl}_3 \oplus \mathfrak{gl}_3)$	3	$P_{3;\mathbb{C}}$
$\mathcal{J}(3,\mathbb{H}) = \mathrm{Herm}(3,\mathbb{H})$	$15 = 3+3d$	4	$\mathfrak{so}_{12}$	$\mathfrak{gl}_6$	3	$P_{3;\mathbb{H}}$
$\mathcal{J}(3,\mathbb{O}) = \mathrm{Herm}(3,\mathbb{O})$	$27 = 3+3d$	8	E_7	$E_6 \oplus \mathfrak{so}_2$	3	$P_{3;\mathbb{O}}$
$\mathcal{J}(4,\mathbb{R}) = \mathrm{Herm}(4,\mathbb{R})$	$10 = 4+6d$	1	$\mathfrak{sp}_8$	$\mathfrak{gl}_4$	4	$P_{4;\mathbb{R}}$
$\mathcal{J}(4,\mathbb{C}) = \mathrm{Herm}(4,\mathbb{C})$	$16 = 4+6d$	2	$\mathfrak{sl}_8$	$\mathfrak{s}(\mathfrak{gl}_4 \oplus \mathfrak{gl}_4)$	4	$P_{4;\mathbb{C}}$
$\mathcal{J}(4,\mathbb{H}) = \mathrm{Herm}(4,\mathbb{H})$	$28 = 4+6d$	4	$\mathfrak{so}_{16}$	$\mathfrak{gl}_8$	4	$P_{4;\mathbb{H}}$

In Table 4.7 we list all pairs $(\mathfrak{g},\mathfrak{k})$ occurring here (i.e., non-hermitian complex symmetric pairs $(\mathfrak{g},\mathfrak{k})$ with $O_{\min} \cap \mathfrak{g}_{\mathbb{R}} \neq \emptyset$ where $\mathfrak{g}$ is simple) together with the Jordan algebra $\mathcal{J}$ arising from the T-K-K theory and the polynomial function $P \in S^4(\mathfrak{k}_1)$ on $\mathcal{J}_{\mathbb{C}}$ written as the product of the Jordan norms.

Table 4.7 Non-Hermitian pairs $(\mathfrak{g},\mathfrak{k})$ with $O_{\min} \cap \mathfrak{g}_{\mathbb{R}} \neq \emptyset$

$\mathcal{J}$	q	P	$\mathfrak{k}$	$\mathfrak{p}$	$\mathfrak{g}$
$\mathcal{J}(4,\mathbb{R})$	4	$P_{4,\mathbb{R}}$	$\mathfrak{sp}_8$	$\wedge^4_o \mathbb{C}^8$	E_6
$\mathcal{J}(4,\mathbb{C})$	4	$P_{4,\mathbb{C}}$	$\mathfrak{sl}_8$	$\wedge^4 \mathbb{C}^8$	E_7
$\mathcal{J}(4,\mathbb{H})$	4	$P_{4,\mathbb{H}}$	$\mathfrak{so}_{16}$	$\mathbb{C}^{128}$	E_8
$\mathcal{J}(3,\mathbb{R}) \oplus \mathcal{J}(1)$	4	$P_{3,\mathbb{R}} P'_1$	$\mathfrak{sp}_6 \oplus \mathfrak{sl}_2$	$\wedge^3_o \mathbb{C}^6 \otimes \mathbb{C}^2$	F_4
$\mathcal{J}(3,\mathbb{C}) \oplus \mathcal{J}(1)$	4	$P_{3,\mathbb{C}} P'_1$	$\mathfrak{sl}_6 \oplus \mathfrak{sl}_2$	$\wedge^3 \mathbb{C}^6 \otimes \mathbb{C}^2$	E_6
$\mathcal{J}(3,\mathbb{H}) \oplus \mathcal{J}(1)$	4	$P_{3,\mathbb{H}} P'_1$	$\mathfrak{so}_{12} \oplus \mathfrak{sl}_2$	$\mathbb{C}^{32} \otimes \mathbb{C}^2$	E_7
$\mathcal{J}(3,\mathbb{O}) \oplus \mathcal{J}(1)$	4	$P_{3,\mathbb{O}} P'_1$	$E_7 \oplus \mathfrak{sl}_2$	$\mathbb{C}^{56} \otimes \mathbb{C}^2$	E_8
$\mathcal{J}(1) \oplus \mathcal{J}'(1)$	2	$P_1^3 P'_1$	$\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$	$S^3\mathbb{C}^2 \otimes \mathbb{C}^2$	G_2
$\mathcal{J}(2;p) \oplus \mathcal{J}'(2;q)$	4	$P_{2;p} P'_{2;q}$	$\mathfrak{so}_p \oplus \mathfrak{so}_q$	$\mathbb{C}^p \otimes \mathbb{C}^q$	$\mathfrak{so}_{p+q}$
$\mathcal{J}(2;p) \oplus \mathcal{J}(1) \oplus \mathcal{J}'(1)$	4	$P_{2;p} P_1 P'_1$	$\mathfrak{so}_p \oplus \mathfrak{so}_4$	$\mathbb{C}^p \otimes \mathbb{C}^4$	$\mathfrak{so}_{p+4}$
$\mathcal{J}(1) \oplus \mathcal{J}'(1) \oplus \mathcal{J}''(1) \oplus \mathcal{J}'''(1)$	4	$P_1 P'_1 P''_1 P'''_1$	$\mathfrak{sl}_2^{\oplus 4}$	$(\mathbb{C}^2)^{\otimes 4}$	$\mathfrak{so}_8$
$\mathcal{J}(2;p) \oplus \mathcal{J}(1)$	3	$P_{2;p} P_1^2$	$\mathfrak{so}_p \oplus \mathfrak{so}_3$	$\mathbb{C}^p \otimes \mathbb{C}^3$	$\mathfrak{so}_{p+3}$
$\mathcal{J}(1) \oplus \mathcal{J}'(1) \oplus \mathcal{J}''(1)$	3	$P_1^2 P'_1 P''_1$	$\mathfrak{so}_3 \oplus \mathfrak{so}_4$	$\mathbb{C}^3 \otimes \mathbb{C}^4$	$\mathfrak{so}_7$
$\mathcal{J}(2;p)$	2	$P_{2;p}^2$	$\mathfrak{so}_p$	$S^2_o \mathbb{C}^p$	$\mathfrak{sl}_p$
$\mathcal{J}(1) \oplus \mathcal{J}'(1)$	2	$P_1^2 {P'_1}^2$	$\mathfrak{so}_3 \oplus \mathfrak{so}_3$	$\mathbb{C}^3 \otimes \mathbb{C}^3$	$\mathfrak{so}_6$
$\mathcal{J}(1)$	1	P_1^4	$\mathfrak{sl}_2$	$S^4\mathbb{C}^2$	$\mathfrak{sl}_3$

$p, q \geq 5$ throughout the table

Comparing Tables 4.6 and 4.7, we find

Proposition 4.8. *There is a bijection between* (i) *the pairs* $(\mathfrak{g},\mathfrak{k})$ *in Table* 4.7 *and* (ii) *the triples* $(\mathcal{J}, P)$ *where* $\mathcal{J}$ *is a Euclidean real Jordan algebra and* P *is a monomial in the Jordan*

norms $P_{[n]}$ of the simple components of $\mathcal{J}$ such that each $P_{[n]}$ occurs at least once in P and P has total degree 4.

Notice that the condition that $\mathcal{J}$ has degree ≤ 4 is necessary but not sufficient for $\mathcal{J}$ to occur here (indeed $\mathcal{J}$ cannot be a Jordan algebra of rank 3) and that the same $\mathcal{J}$ can give rise to different polynomials P and hence different $\mathfrak{g}$ (this occurs for $\mathfrak{g} = G_2$ and $\mathfrak{g} = \mathfrak{so}_6$).

From Table 4.7, we get in Table 4.9 a list of the real semisimple groups $G_{\mathbb{R}}$ occurring here.

Table 4.9.

$G_{\mathbb{R}}$	$\mathfrak{k}$	rank	d	m	
$E_{6(6)}$	$\mathfrak{sp}_8$	6	1	$4+6d=10$	
$E_{7(7)}$	$\mathfrak{sl}_8$	7	2	$4+6d=16$	
$E_{8(8)}$	$\mathfrak{so}_{16}$	8	4	$4+6d=28$	
$F_{4(4)}$	$\mathfrak{sp}_6 \oplus \mathfrak{sl}_2$	4	1	$4+3d=7$	
$E_{6(2)}$	$\mathfrak{sl}_6 \oplus \mathfrak{sl}_2$	4	2	$4+3d=10$	
$E_{7(-5)}$	$\mathfrak{so}_{12} \oplus \mathfrak{sl}_2$	4	4	$4+3d=16$	
$E_{8(-24)}$	$E_7 \oplus \mathfrak{sl}_2$	4	8	$4+3d=28$	
$G_{2(2)}$	$\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$	2	$\frac{2}{3}$	$(4+3d)/3=2$	
$\widetilde{SO}(p,q)$	$\mathfrak{so}_p \oplus \mathfrak{so}_q$	p		$p+q-4$	$3 \leq p \leq q$
$\widetilde{SL}(n,\mathbb{R})$	$\mathfrak{so}_n$	$n-1$		$n-2$	$3 \leq n$

In the listing of the exceptional real groups, the subscripted number in parentheses is equal to $\dim_{\mathbb{R}} \mathfrak{p}_{\mathbb{R}} - \dim_{\mathbb{R}} \mathfrak{k}_{\mathbb{R}}$ and serves to distinguish between simply-connected real forms having the same complexified Lie algebra. In the first three cases d is the "correct" parameter for the corresponding simple Jordan algebra $\mathfrak{k}_{-1}$ while in the next five cases d is a fictitious parameter which we make up as it gives consistent formulas in Tables 4.9 and 6.9.

§5. Holomorphic Half-Form Bundles on $O_{\mathbb{R}}$.

In this section, we construct and classify all holomorphic half-form bundles $\mathbf{N}^{\frac{1}{2}}$ over $O_{\mathbb{R}}$ equipped with its instanton Kaehler structure $\mathbf{J}$ from §2. Right away, we identify $(O_{\mathbb{R}}, \mathbf{J})$ with the complex cone Y in $\mathfrak{p}$ by means of the Vergne diffeomorphism (2.3).

We are assuming, throughout the rest of the paper, that $O_{\mathbb{R}}$ is strongly minimal and $(\mathfrak{g}, \mathfrak{k})$ is non-hermitian. Then $\mathfrak{k}$ is a semisimple Lie algebra and $\mathfrak{p}$ is irreducible as a representation of K. The spaces $O_{\mathbb{R}}$ and Y are given by (3.7). In particular Y is the conical K-orbit of highest weight vectors in $\mathfrak{p}$. The cases occurring here were classified in Table 4.7.

Each holomorphic half-form bundle $\mathbf{N}^{\frac{1}{2}}$ over Y is automatically homogeneous under K (see Lemma 5.2). The space $H = \Gamma(Y, \mathbf{N}^{\frac{1}{2}})$ of global algebraic holomorphic sections breaks up under the action of K into a multiplicity free ladder decomposition which we analyze in

Lemmas 5.2 and 5.3.

In Proposition 5.5 we determine the spectrum of the operator E' on H given by the Lie derivative of the holomorphic Euler vector field E. We find that E' is diagonalizable with *positive* spectrum. This result about E' is crucial since it allows us to *invert* E' and $E'+1$ in §6 in order to quantize the symbol g_{v_0} from Theorem 3.6.

We regard E' as the "energy" operator on the space H of quantization. Indeed, $E' = \mathcal{Q}(\lambda) = \mathcal{Q}(\rho)$ by Corollary 2.2(i). I.e., E' is the quantization of our chosen Hamiltonian ρ (see discussion before (2.8)). Since ρ is positive everywhere on $O_{\mathbb{R}}$, the positivity of E' is exactly what we expect from the quantum theory.

The Euler $\mathbb{C}^*$-action on Y (see §2) defines a complex algebra grading

$$R(Y) = \bigoplus_{p\in\mathbb{Z}_+} R_p(Y) \tag{5.1}$$

where

$$R_p(Y) = \{f \in R(Y) \,|\, Ef = pf\} \tag{5.2}$$

Here $\mathbb{Z}_+$ denotes the set of non-negative integers. A priori, the grading in (5.1) extends over all integers, but since Y is the orbit of highest weight vectors in $\mathfrak{p}$, it follows that the pullback map $S(\mathfrak{p}^*) \to R(Y)$ on functions is surjective. So $R_p(Y) = 0$ for p negative.

Our first aim is to compute the fundamental group of Y. To do this, we use fact that Y is a homogeneous space of K:

$$Y \simeq K/K^e \tag{5.3}$$

Since $\pi_1(K) = 0$, it follows that $\pi_1(Y)$ is isomorphic to the component group of K^e.

Now let $Q \subset K$ be the (closed) subgroup which preserves the line $\mathbb{C}e$. Clearly K^e lies in Q as the kernel of the action of Q on $\mathbb{C}e$; we put $Q' = K^e$. The quotient $\mathbb{P}(Y)$ of Y by the Euler $\mathbb{C}^*$-action identifies with K/Q. Since Y is an orbit of highest weight vectors, $\mathbb{P}(Y)$ is a (generalized) flag variety of K. So we have a K-equivariant principal $\mathbb{C}^*$-bundle

$$Y \;\to\; \mathbb{P}(Y) \;\simeq\; K/Q \tag{5.4}$$

It follows easily that Q is the connected subgroup of K with Lie algebra $\mathfrak{q} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$. Thus we get the Levi decomposition

$$Q = K_0 \ltimes K_1 \tag{5.5}$$

where $K_1 = \exp \mathfrak{k}_1$ is the connected unipotent subgroup of K with Lie algebra $\mathfrak{k}_1$.

Lemma 5.1. *Y is simply-connected except if $\mathfrak{g}$ is of type A_n. If $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, then the fundamental group of Y is $\pi_1(Y) \simeq \mathbb{Z}_2$ if $n \geq 4$ or $\pi_1(Y) \simeq \mathbb{Z}_4$ if $n = 3$.*

Proof. The discussion above gives $K^e = K_0' \ltimes K_1$ where K_0' was defined in (4.3). The exponential map $\exp : \mathfrak{k}_1 \to K_1$ is an isomorphism. Hence, the component groups of K^e and

K_0' identify. So $\pi_1(Y)$ is isomorphic to the component group of $K_0' = \operatorname{Ker}\chi$ where χ is the K_0-weight of e.

Proposition 4.3 says that χ is the square root of the weight of P. (The square root is unique since K_0 is connected.) Thus the product decomposition (4.13) gives $\chi^2 = \chi_1^{2w_1} \cdots \chi_\ell^{2w_\ell}$ where χ_n^2 is the weight of the Jordan norm $P_{[n]}$. So

$$\chi = \chi_1^{w_1} \cdots \chi_\ell^{w_\ell} \tag{5.6}$$

The well-known theory of the K_0-action on $S(\mathfrak{k}_1)$ (see [B-K3, Theorem 3.4 and Corollary 3.6]) says that $\chi_1, \ldots, \chi_\ell$ are primitive characters of K_0 and form a basis of the character group. The kernel of a primitive character is connected. It follows that the component group of $\operatorname{Ker}\chi$ is isomorphic to $\mathbb{Z}/g\mathbb{Z}$ where $g = \gcd\{w_1, \ldots, w_\ell\}$. Now the Lemma follows immediately from Table 4.7 where P is given explicitly in the third column as a monomial in $P_{[1]}, \ldots, P_{[\ell]}$ so that the exponents $w_1, \ldots, w_\ell$ can be read off in each case. $\blacksquare$

It is often convenient to label irreducible representations of K by their highest weight in the sense of the Cartan-Weyl highest weight theory. This requires that we fix a choice of Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{k}$ together with a Borel subalgebra $\mathfrak{b}$ in $\mathfrak{k}$. The set $\mathcal{R}^+$ of non-zero weights of $\mathfrak{h}$ on $\mathfrak{b}$ is the set of *positive roots*. Then $\mathcal{R} = \mathcal{R}^+ \cup -\mathcal{R}^+$ is the set of all non-zero weights of $\mathfrak{h}$ on $\mathfrak{g}$. Each $\alpha \in \mathcal{R}$ is called a *root* and the corresponding weight space $\mathfrak{k}^\alpha$ is 1-dimensional and is called the α-root space. Then

$$\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m} \oplus \mathfrak{m}^- \tag{5.7}$$

where $\mathfrak{m} \subset \mathfrak{b}$ is the span of the positive root spaces and $\mathfrak{m}^-$ is the span of the negative root spaces.

Cartan-Weyl highest weight theory says that in each finite-dimensional irreducible $\mathfrak{k}$-representation V, the subspace $V^{\mathfrak{m}}$ of all vectors annihilated by the action of $\mathfrak{m}$ is 1-dimensional. Clearly then $V^{\mathfrak{m}}$ is a weight space of $\mathfrak{h}$ of some weight μ. Moreover $V^{\mathfrak{m}}$ turns out to be the full μ-weight space V^μ in V. Then μ is the so-called *highest weight* in V, any (non-zero) vector in V^μ is called a *highest weight vector*, and we write $V = V_\mu$.

We choose $(\mathfrak{h}, \mathfrak{b})$ so that $h \in \mathfrak{h}$ and e is a highest weight vector in $\mathfrak{p}$. We also require that $\mathfrak{k}_1$ lies in $\mathfrak{m}$, $\mathfrak{h}$ is complex conjugation stable and $\mathfrak{h}$ is stable under the complex Cartan involution of the symmetric pair $(\mathfrak{k}_0, \mathfrak{k}^{\mathfrak{l}})$. It is easy to meet these conditions. Then

$$\mathfrak{p} \simeq V_\psi \tag{5.8}$$

where ψ is the weight of e so that ψ is the highest weight of $\mathfrak{p}$. Notice that ψ is just the restriction to $\mathfrak{h}$ of the $\mathfrak{k}_0$-weight $\psi = \mathrm{d}\chi \in \mathfrak{k}_0^*$.

From now on in this section, we work in the category of algebraic holomorphic complex line bundles L over a smooth (irreducible) complex algebraic manifold X, which will soon be specialized to $X = Y$. $R(X)$ and $\Gamma(X, L)$ denote the regular (i.e., algebraic holomorphic) functions on X and sections of L.

Next we discuss some general notions about half-form bundles. Let $L \to X$ be a complex line bundle. A *square root* of L is a pair (C, α) where C is a complex line bundle over X and $\alpha : C^{\otimes 2} \to L$ is a bundle isomorphism. Notice that α gives an $R(X)$-module map

$$\alpha_X : \Gamma(X, C)^{\otimes 2} \to \Gamma(X, L)$$

so that the product of two sections of C defines a section of L.

Two square roots (C, α) and (C', α') are isomorphic if there exists a bundle isomorphism $\beta : C \to C'$ such that $\alpha = \alpha' \circ \beta^{\otimes 2}$ where $\beta^{\otimes 2} : C^{\otimes 2} \to (C')^{\otimes 2}$ is the obvious map. In counting or classifying half-form bundles, we will always work up to isomorphism.

It is easy to check that if $s \in \Gamma(X, L)$ is a non-zero section then there exists, up to isomorphism, at most one square root (C, α) of L such that s is the square of some section of C. In practice, we suppress the isomorphism α from the notation. Notice that any line bundle is a square root of its square.

We will use the notation and terminology from [B-K3] on algebraic holomorphic differential operators and their symbols. If $\eta \in \mathcal{D}^1(X, L)$, i.e., η is an order 1 differential operator on sections of L, then η determines an order 1 differential operator on sections of any square root C of L in the following way: there exists a unique operator $\eta^\sharp \in \mathcal{D}^1(X, C)$ such that $\eta(s^2) = 2s\eta^\sharp(s)$ for all $s \in \Gamma(X, C)$. Then the symbols of η and $\eta^\sharp$ coincide. We write η for $\eta^\sharp$ when the meaning is clear.

The next result follows easily using general facts about homogeneous line bundles for the first part and [B-K3, Lem. 2.9 and Appendix §A.12] for the second part. In particular, the proof of the second part uses the Borel-Weil theorem on $\mathbb{P}(Y)$.

Recall that $f_0 \in R(Y)$ was defined in (3.15). Using the terminology of this section we see that f_0 is a highest weight vector in $R_1(Y)$ of weight ψ and f_0 is Q-semi-invariant.

Lemma 5.2. *Suppose C is a square root of a K-homogeneous line bundle on Y. Then C has (uniquely) the structure of a K-homogeneous line bundle. The space $H = \Gamma(Y, C)$ of global sections is non-zero. The differential of the corresponding K-action on H is a representation of $\mathfrak{k}$ on H by differential operators of order 1 and is compatible with tensor product of bundles.*

The $\mathfrak{k}$-representation on H is completely reducible and multiplicity-free. A vector in H is a highest weight vector for the $\mathfrak{k}$-action if and only if it is Q-semi-invariant. The space $H^{\mathfrak{m}}$ of highest weight vectors has a basis of the form $\{f_0^p s_0 \,|\, p \in \mathbb{Z}_+\}$ where $s_0 \in H$ is uniquely determined up to scaling. This gives a ladder decomposition

$$H \simeq \bigoplus_{p \in \mathbb{Z}_+} V_{\nu + p\psi} \tag{5.9}$$

of H as a $\mathfrak{k}$-representation where ν is the $\mathfrak{h}$-weight of s_0.

We write $H_{[\nu+p\psi]}$ for the subspace of H which carries the $\mathfrak{k}$-representation $V_{\nu+p\psi}$. The representation V_ν is the minimal $\mathfrak{k}$-type in H. We may also call its carrier space $H_{[\nu]}$ the minimal $\mathfrak{k}$-type in H.

A square root $\mathbf{N}^{\frac{1}{2}}$ of the canonical line bundle $\mathbf{N} = \wedge^n(T^*X)$ on X, where $n = \dim_{\mathbb{C}} X$, is called an algebraic holomorphic *half-form* line bundle on X. Sections of $\mathbf{N}^{\frac{1}{2}}$ are called *half-forms*. Let $\mathcal{L}_\xi$ denote the Lie derivative of a vector field ξ on X so that in particular we get

an operator $\mathcal{L}_\xi \in \mathcal{D}^1(X,\mathbf{N})$. The Lie derivative on half-forms is the operator $\mathcal{L}_\xi \in \mathcal{D}^1(X,\mathbf{N}^{\frac{1}{2}})$ given by

$$\mathcal{L}_\xi(s^2) = 2s\mathcal{L}_\xi(s) \tag{5.10}$$

In this way we get a representation

$$\mathfrak{V}ect(X) \to \mathcal{D}(X,\mathbf{N}^{\frac{1}{2}}) \to \operatorname{End} H, \qquad \xi \mapsto \mathcal{L}_\xi \tag{5.11}$$

where $\mathfrak{V}ect(X)$ is the Lie algebra of algebraic holomorphic vector fields on X. Then on the symbol level

$$\text{symbol}\, \mathcal{L}_\xi \;=\; \text{symbol}\, \xi \,\in R_{[1]}(T^*X) \tag{5.12}$$

where $R_{[p]}(T^*X)$ denotes the space of regular functions on the cotangent bundle T^*X which are homogeneous of degree p on the fibers of the natural projection $T^*X \to X$.

Now the Lie derivative of the Euler vector field E on Y is the operator

$$E' = \mathcal{L}_E \tag{5.13}$$

The following result is easy to verify and defines the minimal E'-eigenvalue r_0.

Lemma 5.3. *Suppose $\mathbf{N}^{\frac{1}{2}}$ is a half-form bundle on Y. Then the K-action on $\mathbf{N}^{\frac{1}{2}}$ gives a natural representation of K on the space $H = \Gamma(Y,\mathbf{N}^{\frac{1}{2}})$ of global sections. Differentiating this gives the representation*

$$\pi'_K : \mathfrak{k} \to \mathcal{D}(Y,\mathbf{N}^{\frac{1}{2}}) \to \operatorname{End} H, \qquad x \mapsto \mathcal{L}_{\eta^x} \tag{5.14}$$

E' is diagonalizable on H with spectrum $r_0 + \mathbb{Z}_+$ where $r_0 \in \frac{1}{2}\mathbb{Z}$. This defines r_0. Thus we have the eigenspace decomposition

$$H = \bigoplus_{p\in\mathbb{Z}_+} H_{r_0+p} \tag{5.15}$$

where H_q is the q-eigenspace of E' in H. The action of E' on H commutes with the K-action and the eigenspaces H_{r_0+p} are the irreducible K-submodules in H. Furthermore H_{r_0+p} carries the representation $V_{\nu+p\psi}$ so that

$$H_{r_0+p} = H_{[\nu+p\psi]} \tag{5.16}$$

Let $s_0 \in H_{r_0}$ be a non-zero Q-semi-invariant section. Then $\{f_0^p s_0 \in H_{r_0+p} \,|\, p \in \mathbb{Z}_+\}$ is a complete set of linearly independent Q-semi-invariant sections in H.

The Hilbert space of our quantization of $O_\mathbb{R}$ will be a certain completion of $H = \Gamma(Y,\mathbf{N}^{\frac{1}{2}})$. We think of E' as the energy operator and call H_{r_0} the *vacuum space* in H. The sections in H_{r_0} are the *vacuum vectors* in H. The vector s_0 chosen in H_{r_0} is unique up to scaling.

A *spherical* K-representation is one that contains a non-zero K-invariant vector, which is then called the spherical vector. It follows from Lemma 5.3 that H is K-spherical if and only if $H_{r_0} \simeq \mathbb{C}$, so if and only if s_0 is K-invariant.

Remark 5.4. Our discussion of square root bundles and half-form bundles generalizes in the obvious way to nth root bundles and nth roots of the canonical bundle. Then all the results in Lemmas 5.2 and 5.3 go over to the case where we replace $\mathbf{N}^{\frac{1}{2}}$ by an nth root of the canonical bundle, the only change being that then $r_0 \in \frac{1}{n}\mathbb{Z}$ instead of $r_0 \in \frac{1}{2}\mathbb{Z}$.

Our key result on half-form bundles is

Proposition 5.5. *Suppose $\mathbf{N}^{\frac{1}{2}}$ is a half-form bundle on Y. Then the minimal eigenvalue r_0 of E' on $H = \Gamma(Y, \mathbf{N}^{\frac{1}{2}})$ is positive. Thus E' has positive spectrum $r_0 + \mathbb{Z}_+$ on H.*

For the proof, we need to construct a concrete holomorphic $(m+1)$-form Λ on Y. We will use this form Λ throughout §5 and §7.. We construct Λ out of the functions $f_0, f_1 \dots, f_m$ on Y we defined in (3.15) in the following way:

$$\Lambda = \mathrm{d}f_0 \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_m \in \Gamma(Y, \mathbf{N}) \tag{5.17}$$

Since $f_0, f_1 \dots, f_m$ were coordinates on the open set Y^o defined in (3.17), it follows that Λ is nowhere vanishing on Y^o. Furthermore Λ is Q-semi-invariant and

$$\zeta = \ Q\text{-weight of } \Lambda \tag{5.18}$$

is the character by which Q acts on $\mathfrak{p}_2 \otimes \wedge^m \mathfrak{p}_1$.

Proof of Proposition 5.5. We will show that the spectrum of $\mathcal{L}_E$ on $\Gamma(Y, \mathbf{N})$ is positive. This implies the positivity of the spectrum of E' on H.

It follows by Lemma 5.2 that $\Gamma(Y, \mathbf{N})$ is a multiplicity-free K-module of ladder type and the set $\{f_0^{p-\beta}\Lambda \,|\, p \in \mathbb{Z}_+\}$ is a complete set of linearly independent highest weight vectors, where β is some non-negative integer. Let

$$\Lambda_0 = f_0^{-\beta}\Lambda \tag{5.19}$$

Since $E(f_v) = f_v$ for all $v \in \mathfrak{p}$ and $\mathcal{L}_{\eta^h}(f_v) = k f_v$ if $v \in \mathfrak{p}_k$, we find

$$\begin{aligned} \mathcal{L}_E(\Lambda_0) =& t\Lambda_0 \quad \text{with} \quad t = -\beta + 1 + m \\ \mathcal{L}_{\eta^h}(\Lambda_0) =& j\Lambda_0 \quad \text{with} \quad j = -2\beta + 2 + m \end{aligned} \tag{5.20}$$

But $j \geq 0$. This follows from the representation theory of the Lie algebra $\mathfrak{l}$ defined in (4.7) as $\mathcal{L}_{\eta^x}(\Lambda_0) = 0$ for all $x \in \mathfrak{k}_1$ and so in particular for $x = e_{\mathfrak{k}}$. Therefore Λ_0 is the highest weight vector of a finite-dimensional irreducible $\mathfrak{l}$-representation. So $2t = j + m$ is positive and hence t is positive. This gives the result as $t + \mathbb{Z}_+$ is the spectrum of E' on $\Gamma(Y, \mathbf{N})$. ■

Proposition 5.6.

(i) *If* $\mathfrak{g}$ *is not of type* A_n*, then up to isomorphism* Y *admits at most one half-form bundle.*
(ii) *If* $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ $(n \geq 3)$*, then either* Y *admits no half-form bundle or* Y *admits exactly two non-isomorphic half-form bundles.*
(iii) *If* $\mathbf{N}^{\frac{1}{2}}$ *is a half-form bundle on* Y *with* $H = \Gamma(Y, \mathbf{N}^{\frac{1}{2}})$ *and* $s_0 \in H_{r_0}$ *is a non-zero* Q*-semi-invariant vector, then up to scaling*

$$\textit{either} \qquad s_0^2 = \Lambda_0 \qquad \textit{or} \qquad s_0^2 = f_0\Lambda_0 \tag{5.21}$$

Moreover the two possibilities in (5.21) *classify half-form bundles on* Y *up to isomorphism.*

Proof. From formal properties of square root bundles, it follows that if the canonical bundle $\mathbf{N}$ on Y admits any square root, then the set of all square roots up to isomorphism is parameterized by the order 2 characters of $\pi_1(Y)$. Thus by Lemma 5.1, if Y admits a half-form bundle then it is unique with the exception of the cases $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ where there would be two half-form bundles. This proves (i) and (ii).

Now suppose we are given $\mathbf{N}^{\frac{1}{2}}$ and s_0. Recall Λ_0 from (5.19). The ladder structure on $\Gamma(Y,\mathbf{N})$ (Lemma 5.2) implies that (up to scaling) $s_0^2 = f_0^p\Lambda_0$ where $p \in \mathbb{Z}_+$. If $p \geq 2$ then we consider the local section $f_0^{-1}s_0 \in \Gamma(Y^o, \mathbf{N}^{\frac{1}{2}})$. The square of $f_0^{-1}s_0$ is equal to $f_0^{p-2}\Lambda_0$ and this has no poles on Y. It follows that $f_0^{-1}s_0$ has no poles on Y and so $f_0^{-1}s_0 \in \Gamma(Y, \mathbf{N}^{\frac{1}{2}})$. But $f_0^{-1}s_0$ has E'-degree equal to $r_0 - 1$ and so this contradicts the minimality of r_0. Thus $p = 0$ or $p = 1$. Furthermore, Λ_0 and $f_0\Lambda_0$ cannot both be squares of sections of the same bundle $\mathbf{N}^{\frac{1}{2}}$ since, by (5.1), $f_0 \in R_1(Y)$ is not a square in $R(Y)$. ■

Proposition 5.7.
(i) *If* $G_{\mathbb{R}} = \widetilde{SO}(p,q)$ *where* $p+q$ *is odd and* $p, q \geq 4$*, then* Y *admits no half-form bundle.*
(ii) *If* $G_{\mathbb{R}} = \widetilde{SL}(p,\mathbb{R})$ *where* p *is odd and* $5 \leq p$*, then* Y *admits no half-form bundle.*
(iii) *If* $G_{\mathbb{R}} = \widetilde{SL}(n,\mathbb{R})$ *where* $n \geq 4$ *is even, then* Y *admits exactly two half-form bundles* $\mathbf{N}_+^{\frac{1}{2}}$ *and* $\mathbf{N}_-^{\frac{1}{2}}$*. These may be characterized by the conditions:* Λ_0 *is the square of a section of* $\mathbf{N}_+^{\frac{1}{2}}$ *and* $f_0\Lambda_0$ *is the square of a section of* $\mathbf{N}_-^{\frac{1}{2}}$*.*
(iv) *In all other cases,* Y *admits a unique half-form bundle* $\mathbf{N}^{\frac{1}{2}}$ *and* Λ_0 *is the square of a section of* $\mathbf{N}^{\frac{1}{2}}$*.*

All the half-form bundles occurring here are listed in Table 6.9 *along with their minimal* E'*-eigenvalue* r_0 *and the* $\mathfrak{k}$*-type* V_ν *of the vacuum space* H_{r_0}*.*

Proof. By (5.19) the form Λ_0 is K_0-semi-invariant of weight $\zeta_0 = \chi^{-\beta}\zeta$. It follows easily that Y admits a half-form bundle with $s_0^2 = \Lambda_0$ (respectively $s_0^2 = f_0\Lambda_0$) if and only if ζ_0 (respectively $\chi\zeta_0$) is the square of a character τ of K_0. Then τ is unique and $d\tau = \nu$ is the highest weight of H_{r_0}.

Thus we need to compute the character ζ_0 and look for square roots in the character group of K_0. The Jordan structure on $\mathfrak{k}_{-1}$ gives us a convenient way to do the calculations. Indeed we found in the proof of Lemma 5.1 (i) the basis $\chi_1, \ldots, \chi_\ell$ of the character group of K_0 and (ii) the formula (5.6).

Now by (5.17) Λ transforms in the 1-dimensional K_0-representation

$$\mathfrak{p}_2 \otimes \wedge^m \mathfrak{p}_1 \;\simeq\; \chi^{m+1} \otimes \wedge^m \mathfrak{k}_{-1} \tag{5.22}$$

But $\wedge^m \mathfrak{k}_{-1}$ is the tensor product of the top exterior powers of the spaces $\mathfrak{j}_{[1]}, \ldots, \mathfrak{j}_{[\ell]}$. The weight of K_0 on the top exterior power of $\mathfrak{j}_{[n]}$ is $\chi_n^{-u_n}$ where $u_n = 2 + d_n(q_n - 1)$. Here d_n is the root multiplicity parameter of $\mathfrak{j}_{[n]}$ given in Table 4.6 and the formula for u_n is immediate from the description of the restricted root system of $(\mathfrak{k}_0, \mathfrak{k}^{\mathfrak{l}})$. Thus the weight of Λ is by (5.6)

$$\zeta = \chi^{m+1}\chi_1^{-u_1} \cdots \chi_\ell^{-u_\ell} = \chi_1^{(m+1)w_1 - u_1} \cdots \chi_\ell^{(m+1)w_\ell - u_\ell} \tag{5.23}$$

We can decide if a section $s = f_0^p \Lambda$ over Y^o extends to Y just by examining its weight $\chi_1^{t_1} \cdots \chi_\ell^{t_\ell}$. Indeed s extends to Y if and only if $t_1, \ldots, t_\ell \geq 0$. This follows easily by using the Borel-Weil theorem on $\mathbb{P}(Y)$ (in its geometric form involving the orders of poles along irreducible divisors in the complement of the big cell) to compute $\Gamma(Y, \mathbf{N})$. Thus, going back to the definition of $\Lambda_0 = f_0^{-\beta}\Lambda$ in (5.19), we see that β is the largest non-negative integer such that all the numbers $-\beta w_n + (m+1)w_n - u_n$ are non-negative. To simplify, we put $\alpha = m + 1 - \beta$. Then the weight of Λ_0 is

$$\zeta_0 = \chi_1^{\alpha w_1 - u_1} \cdots \chi_\ell^{\alpha w_\ell - u_\ell} \tag{5.24}$$

where α is the smallest positive integer such that $\alpha w_n \geq u_n$ for all n.

It is now easy to go through Table 4.7, calculate ζ_0 in each case, and see if ζ_0 and/or $\chi\zeta_0$ admits a square root. The cases group together naturally into families. In the first three cases in Table 4.7 we have $P = P_{4,\mathbb{R}}, P_{4,\mathbb{C}}, P_{4,\mathbb{H}}$ so that $\ell = 1$, $w_1 = 1$, $u_1 = 2 + 3d$. Then $\alpha = 2 + 3d$ and $\zeta_0 = 1$. Thus we get a half-form bundle (unique as $\mathfrak{g} \neq \mathfrak{sl}(p, \mathbb{C})$) with $s_0^2 = \Lambda_0$ and $H_{r_0} \simeq \mathbb{C}$.

In the four cases where $P = P_{3,\mathbb{F}}P_1'$, we have $\ell = 2$, $(w_1, w_2) = (1, 1)$ and $(u_1, u_2) = (2 + 2d, 2)$ where $d = d_1$. Then $\alpha = 2 + 2d$ and $\zeta_0 = \chi_2^{2d}$ which is a square. So we get a unique half-form bundle with $s_0^2 = \Lambda_0$ and $H_{r_0} \simeq \mathbb{C} \otimes S^d\mathbb{C}^2$. The G_2 case $P = P_1^3 P_1'$ is similar with $(w_1, w_2) = (3, 1)$ and $(u_1, u_2) = (1, 1)$. Then $\alpha = 2$ and $\zeta_0 = \chi_1^4$ which is a square. This gives a unique half-form bundle with $s_0^2 = \Lambda_0$ and we find $H_{r_0} \simeq S^2\mathbb{C}^2 \otimes \mathbb{C}$ since $\chi = \chi_1^2\chi_2$ is the highest weight of $\mathfrak{p} = S^3\mathbb{C}^2 \otimes \mathbb{C}^2$.

We can treat all the cases where $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(p, q)$, $3 \leq p \leq q$, simultaneously with $\ell = 2$ as long as we formally set $\mathcal{J}(2;4) = \mathcal{J}(1) \oplus \mathcal{J}(1)$, $P_{2;4} = P_1 P_1'$, and $\mathcal{J}(2;3) = \mathcal{J}(1)$, $P_{2;3} = P_1^2$ with $S^{k/2}\mathbb{C}^3 = S^k\mathbb{C}^2$ as $\mathfrak{so}(3)$-representations. This follows easily and we get $(w_1, w_2) = (1, 1)$ and $(u_1, u_2) = (p - 2, q - 2)$. Then $\alpha = q - 2$ and $\zeta_0 = \chi_1^{q-p}$. If $q - p$ is even, then ζ_0 is a square, and we get a half-form bundle with $s_0^2 = \Lambda_0$ and $H_{r_0} \simeq S^{(q-p)/2}\mathbb{C}^2 \otimes \mathbb{C}$ since $\chi = \chi_1\chi_2$ is the highest weight of $\mathfrak{p} \simeq \mathbb{C}^p \otimes \mathbb{C}^q$. This is the unique half-form bundle unless $(p, q) = (3, 3)$ in which case we get a second half-form bundle with $s_0^2 = f_0\Lambda_0$ and $H_{r_0}^{\triangle 2} \simeq \mathfrak{p}$ (where $\triangle$ denotes Cartan product) so that $H_{r_0} \simeq \mathbb{C}^2 \otimes \mathbb{C}^2$. Indeed $\mathfrak{so}(3,3) = \mathfrak{sl}(4, \mathbb{R})$ and this is the only time when our $\mathfrak{so}(p, q)$ cases and $\mathfrak{sl}(n, \mathbb{R})$ cases coincide. Now if $q - p$ is odd with $p > 3$, then neither ζ_0 nor $\chi\zeta_0 = (\chi_1^{q-p+1}, \chi_2)$ are squares. However, if $q - p$ is odd with

$p=3$, then ζ_0, but not $\chi\zeta_0$, is a square. Thus we get one half-form bundle with $s_0^2=\Lambda_0$ and $H_{r_0}\simeq S^{(q-3)/2}(\mathbb{C}^3)\otimes\mathbb{C}=S^{q-3}\mathbb{C}^2\otimes\mathbb{C}$.

Finally we consider the $\mathfrak{g}_\mathbb{R}=\mathfrak{sl}(p,\mathbb{R})$ cases. If $p\geq 5$, then $P=P^2_{2;p}$, $\ell=1$, $w_1=2$, $u_1=p-2$. So if p is even then $\alpha=(p-2)/2$, ζ_0 is trivial and we get a half-form bundle with $s_0^2=\Lambda_0$ and $H_{r_0}\simeq\mathbb{C}$. The second half-form bundle has $s_0^2=f_0\Lambda_0$ and $H^{\triangle 2}_{r_0}\simeq\mathfrak{p}$ so that $H_{r_0}\simeq\mathbb{C}^p$. If p is odd, then there is no half-form bundle. We already did the case $p=4$. If $p=3$ then $P=P^4_1$, $\ell=1$, $w_1=4$, $u_1=2$ and so $\alpha=1$ and $\zeta_0=\chi_1^2$. We get one half-form bundle with $s_0^2=\Lambda_0$ and $H_{r_0}\simeq\mathbb{C}^2$, and a second with $s_0^2=f_0\Lambda_0$ and $H_{r_0}\simeq S^3\mathbb{C}^2$.

We have now proven everything except for the values of r_0. But (5.19) and (5.20) give $E'\Lambda_0=(-\beta+m+1)\Lambda_0=\alpha\Lambda_0$. So if $s_0^2=\Lambda_0$ then $r_0=\alpha/2$ while if $s_0^2=f_0\Lambda_0$ then $r_0=(\alpha+1)/2$. We have computed the parameter α in each case above, and this produces the values of r_0 in Table 6.9. ■

Remark 5.8. Case (i) is the *Howe-Vogan counterexample.* Howe proved that these groups $SO(p,q)$ admit no minimal unitary representation and then Vogan ([Vo1]) extended this to the simply-connected covering groups.

The isomorphism (3.16) implies in particular that $R(Y^o)$ is the localization of $R(Y)$ at f_0 so that

$$R(Y^o)=R(Y)[f_0^{-1}]=\mathbb{C}[f_0,f_1,\ldots,f_m][f_0^{-1}] \tag{5.25}$$

It is easy to prove

Proposition 5.9. *Suppose C is a K-homogeneous line bundle on Y and $s\in\Gamma(Y,C)$ is a non-zero Q-semi-invariant vector. Then s is nowhere vanishing on Y^o. Consequently, since Y^o is affine, the space of sections $\Gamma(Y^o,C)$ is a cyclic $R(Y^o)$-module generated by s so that*

$$\Gamma(Y^o,C)=\Gamma(Y,C)[f_0^{-1}]=R(Y^o)s \tag{5.26}$$

In particular then, in Proposition 5.6, *$H=\Gamma(Y^o,\mathbf{N}^{\frac12})$ is a cyclic $R(Y^o)$-module generated by any section $f_0^p s_0$.*

This is a key result as it enables us to analyze H in a uniform manner in §7 below regardless of whether H is K-spherical or not. In fact, we further simplify our work in §7 by making the following observation, which obviates the need to consider separately the two possibilities in (5.21).

The regular function $f_0\in R_1(Y)$ is not a square in $R(Y^o)$ (because of (5.1) and (5.25)) and is nowhere vanishing on Y^o by the definition of Y^o. Thus we may construct a non-trivial two-fold étale covering

$$\widetilde{Y^o}\to Y^o \tag{5.27}$$

by "extracting a square root of f_0". Then $\widetilde{Y^o}$, like Y^o, is again a smooth affine complex algebraic variety and has a unique K_0-action such that the cover $\widetilde{Y^o}\to Y^o$ is K_0-equivariant.

The pull back of $\mathbf{N}^{\frac{1}{2}}$ through the covering is a K_0-homogeneous bundle on $\widetilde{Y^o}$ which we again call $\mathbf{N}^{\frac{1}{2}}$. Now Proposition 5.9 gives

Corollary 5.10. *The space $\Gamma(\widetilde{Y^o}, \mathbf{N}^{\frac{1}{2}})$ of algebraic holomorphic sections is a cyclic $R(\widetilde{Y^o})$-module generated by $\sqrt{\Lambda}$ so that*

$$\Gamma(\widetilde{Y^o}, \mathbf{N}^{\frac{1}{2}}) = \mathbb{C}[f_0^{\frac{1}{2}}, f_0^{-\frac{1}{2}}, f_1, \ldots, f_m]\sqrt{\Lambda} \tag{5.28}$$

§6. Quantization of $O_{\mathbb{R}}$.

In this section we construct explicitly our quantization of $O_{\mathbb{R}}$. This is purely "from scratch"; we assume no a priori information on the existence of any quantizations or unitary representations. In the next section we prove that the constructions of this section "work", i.e., we prove Theorem 6.3 and 6.8.

The first step of our quantization, carried out in §2 and §3, was to transform the quantization problem on $O_{\mathbb{R}}$ into a quantization problem on T^*Y. We replaced (by holomorphic extension) each function ϕ^w, $w \in \mathfrak{g}_{\mathbb{R}}$, by a rational (pseudo-differential) symbol $\Phi^w \in R(T^*Y)[\lambda^{-1}]$. Then, after complexification, we ended up in (3.6) with a realization of $\mathfrak{g}$ as a complex Lie algebra of rational symbols,

$$\Phi^z \in R(T^*Y)[\lambda^{-1}], \qquad z \in \mathfrak{g} \tag{6.1}$$

Our aim now is to quantize the symbols Φ^z, $z \in \mathfrak{g}$, into operators $\mathcal{Q}(\Phi^z)$ on a Hilbert space $\mathcal{H}$ which is a completion of $H = \Gamma(Y, \mathbf{N}^{\frac{1}{2}})$ for some half-form bundle $\mathbf{N}^{\frac{1}{2}}$ over Y. As usual, we freely identify $(O_{\mathbb{R}}, \mathbf{J})$ with Y via the Vergne diffeomorphism (2.3). We require our operators satisfy certain explicit and implicit axioms. This solves our quantization problem on $O_{\mathbb{R}}$ as we set $\mathcal{Q}(\phi^w) = \mathcal{Q}(\Phi^w)$, $w \in \mathfrak{g}_{\mathbb{R}}$.

In §5 we have already quantized the symbol λ, corresponding to our chosen Hamiltonian ρ (cf. Corollary 2.2(i)), into the operator E' on half-forms.

We require that the operators $\mathcal{Q}(\Phi^w)$, $w \in \mathfrak{g}_{\mathbb{R}}$, be self-adjoint, or equivalently, that the operators $\mathcal{Q}(\Phi^z)$ satisfy

$$\mathcal{Q}(\Phi^z)^\dagger = \mathcal{Q}(\Phi^{\overline{z}}), \qquad z \in \mathfrak{g} \tag{6.2}$$

Of course at this point, H carries no preferred positive definite Hermitian inner product. So we will have to construct this along the way. Our operators will not be defined everywhere on $\mathcal{H}$, but they will all contain H in their domain.

We further require that the Dirac commutation relations (2.4) be satisfied. If we put

$$\pi^z = i\mathcal{Q}(\Phi^z), \qquad z \in \mathfrak{g} \tag{6.3}$$

then these relations amount to

$$[\pi^z, \pi^{z'}] = \pi^{[z,z']} \tag{6.4}$$

for all $z, z' \in \mathfrak{g}$. I.e., the map $\pi : \mathfrak{g} \to \operatorname{End} H$, $z \mapsto \pi^z$, must be a complex Lie algebra homomorphism.

In order to satisfy the implicit axioms, we require that the "symbol" of $\mathcal{Q}(\Phi^z)$ is just Φ^z. Here our definition of "symbol" is not precise, as we found in Theorem 3.1(iii) that the symbols Φ^v, $v \in \mathfrak{p}$ are not homogeneous. However, we will get around this by dealing individually with the homogeneous pieces.

With this in mind, we mandate

$$\pi^x = i\mathcal{Q}(\Phi^x) = \mathcal{L}_{\eta^x}, \qquad x \in \mathfrak{k} \tag{6.5}$$

So π^x is the Lie derivative on half-forms of the algebraic holomorphic vector field η^x defined in (2.6) by differentiating the K-action on Y. Thus, just as we would expect, π^x corresponds to the natural K-action on H, i.e., $\pi^x = \pi'_K(x)$ in the notation of (5.14).

Next, guided by the complex Cartan decomposition (3.5), we need to quantize the symbols Φ^v, $v \in \mathfrak{p}$. In (3.4) we found these break into a sum $f_v + g_v$ of two homogeneous pieces. We now mandate

$$\pi^v = i\mathcal{Q}(\Phi^v) = if_v + iT_v, \qquad v \in \mathfrak{p} \tag{6.6}$$

where $T_v = \mathcal{Q}(g_v)$ is some "nice" quantization of the homogeneous degree 2 rational symbol g_v from (3.4).

This leaves the problem of how to construct T_v. Of course we want in the end for π to be a complex Lie algebra homomorphism. So we certainly want $[\pi^x, \pi^v] = \pi^{[x,v]}$ and this implies

$$[\pi^x, T_v] = T_{[x,v]} \tag{6.7}$$

Hence the fact that $\mathfrak{p}$ is irreducible as a $\mathfrak{k}$-representation insures that T_{v_0} and the π^x, $x \in \mathfrak{k}$, already determine all operators T_v, $v \in \mathfrak{p}$.

To construct T_{v_0}, we break down the symbol $g_e = g_{v_0}$ computed in Theorem 3.6 by (3.25) into its elementary factors. On the face of it, it seems hard to imagine what to do with the factor $f_0^{-1}(\Phi_K P)$ appearing in (3.4). While $\Phi_K P$ is the symbol of the perfectly nice order 4 differential operator $\pi'_K(P)$ on sections of $\mathbf{N}^{\frac{1}{2}}$ because of (4.5) and (5.14), the quotient $f_0^{-1}(\Phi_K P)$ a priori only defines a differential operator on sections of the bundle $\mathbf{N}^{\frac{1}{2}}$ restricted to the open set Y^o from (3.17). Fortunately, the result in [B-K3, Ths. 3.10 and 4.5] (which applies more generally to any homogeneous line bundle over Y) tells us

Theorem 6.1. *Suppose $\mathbf{N}^{\frac{1}{2}}$ is a half-form bundle on Y and $H = \Gamma(Y, \mathbf{N}^{\frac{1}{2}})$. Then the operators f_0 and $\pi'_K P$ on H commute and have the same image. Hence the formula*

$$D_e = \frac{1}{f_0}(\pi'_K P) \tag{6.8}$$

defines an operator on H. It follows that D_e is an algebraic differential operator of order 4 *on sections of* $\mathbf{N}^{\frac{1}{2}}$.

The assignment $e \mapsto D_e$ extends naturally and uniquely to a complex linear 1*-to-*1 *K-equivariant map*

$$\mathfrak{p} \to \mathcal{D}(Y, \mathbf{N}^{\frac{1}{2}}), \qquad v \mapsto D_v \tag{6.9}$$

of $\mathfrak{p}$ into the algebra $\mathcal{D}(Y, \mathbf{N}^{\frac{1}{2}})$ of algebraic differential operators on sections of $\mathbf{N}^{\frac{1}{2}}$. Then, for each non-zero $v \in \mathfrak{p}$, D_v has order 4*; also D_v has degree* -1, *i.e.*, $[E', D_v] = -D_v$.

The subalgebra $\mathcal{A} \subset \mathcal{D}(Y, \mathbf{N}^{\frac{1}{2}})$ generated by the D_v, $v \in \mathfrak{p}$, is abelian, isomorphic to $R(Y)$ and graded by $\mathcal{A} = \oplus_{p \geq 0} \mathcal{A}_{-p}$ where $\mathcal{A}_{-p} = \{D \in \mathcal{A} \mid [E', D] = -pD\}$. Putting $D_v = D_{f_v}$ for $v \in \mathfrak{p}$, we get a graded K-equivariant complex algebra isomorphism

$$R(Y) \to \mathcal{A}, \qquad f \mapsto D_f \tag{6.10}$$

There is a unique $K_{\mathbb{R}}$-invariant positive-definite Hermitian inner product B_o on H such that $B_o(s_0, s_0) = 1$ (when we fix a choice of s_0 in Lemma 5.3*) and the operators f_v and $D_{\overline{v}}$ are adjoint with respect to B_o for all $v \in \mathfrak{p}$. Then the grading* (5.15) *is a B_o-orthogonal decomposition.*

The expression for D_e in terms of our local coordinates (3.15) on Y is, in the notation of (3.24),

$$D_e = f_0^3 P\big(\mathcal{L}_{\frac{\partial}{\partial f_1}}, \ldots, \mathcal{L}_{\frac{\partial}{\partial f_m}}\big) \tag{6.11}$$

To complete our quantization of the rational symbol g_e, we need to quantize the factor $-\lambda^{-2}$ in (3.25). It is natural to try to quantize $-\lambda^{-2}$ into the operator $(E' + a)^{-1}(E' + b)^{-1}$ where a and b are some constants to be determined. Of course a and b must be chosen so that neither $-a$ nor $-b$ belongs to the spectrum of E'.

In fact it turns out that exactly one choice, namely $a = 0$ and $b = 1$, satisfies our requirement that the resulting operators defined by (6.5) and (6.6) satisfy the bracket relations of $\mathfrak{g}$. In fact, just the one relation $[\pi^e, \pi^{\overline{e}}] = \pi^h$ mandates that

$$-\frac{1}{\lambda^2} \text{ quantizes to } \frac{1}{E'(E' + 1)} \tag{6.12}$$

(We prove in §7 that this choice works.) We emphasize that the operators E' and $E' + 1$ are invertible since the spectrum of E' on H is positive by Proposition 5.5.

Thus, rather than putting $T_v = \frac{1}{(E'+a)(E'+b)} D_v$ and solving for a and b in §7 we simply define

$$T_v = \mathcal{Q}(g_v) = \frac{1}{E'(E' + 1)} D_v \tag{6.13}$$

In [B2, proof of Theorem 4.2], we wrote out the latter procedure of determining a and b from the bracket relation for the case $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}(3, \mathbb{R})$.

These operators T_v are no longer differential operators, but they share many of the same properties. To explain this, we introduce the notion of $\mathfrak{k}$-finite endomorphism.

We have a natural representation of $\mathfrak{k}$ on $\operatorname{End} H$ defined by $x * D = [\mathcal{L}_{\eta^x}, D]$. Then $D \in \operatorname{End} H$ is called *$\mathfrak{k}$-finite* if D generates a finite-dimensional representation of $\mathfrak{k}$ inside $\operatorname{End} H$. The space $\operatorname{End}_{\mathfrak{k}-fin} H$ of all $\mathfrak{k}$-finite endomorphisms of H is a complex subalgebra of $\operatorname{End} H$. Let

$$\operatorname{End}_{[p]} H \subset \operatorname{End}_{\mathfrak{k}-fin} H \tag{6.14}$$

be the p-eigenspace of $\operatorname{ad} E'$. We have a natural graded K-equivariant complex algebra inclusion (see [B-K4, A.6, A.12]) $\mathcal{D}(Y, \mathbf{N}^{\frac{1}{2}}) \subset \operatorname{End}_{\mathfrak{k}-fin} H$.

Now Theorem 6.1 easily gives (cf. proof of [B-K4, Th. 3.4])

Corollary 6.2. *The operator T_v, $v \in \mathfrak{p}$, lies $\operatorname{End}_{[-1]} H$. Thus we get a K-equivariant complex linear map*

$$T : \mathfrak{p} \to \operatorname{End}_{[-1]} H, \qquad v \mapsto T_v \tag{6.15}$$

The pseudo-differential operators T_v, $v \in \mathfrak{p}$, commute and generate a graded abelian K-stable subalgebra $\mathcal{T} = \oplus_{p \geq 0} \mathcal{T}_{-p}$ of $\operatorname{End}_{\mathfrak{k}-fin} H$ where $\mathcal{T}_{-p} = \mathcal{T} \cap \operatorname{End}_{[-p]} H$. We then get a graded K-equivariant complex algebra isomorphism

$$R(Y) \to \mathcal{T}, \qquad f \mapsto T_f \tag{6.16}$$

where $T_{f_v} = T_v$ for $v \in \mathfrak{p}$. There is a $K_{\mathbb{R}}$-invariant positive-definite Hermitian inner product B on H, such that $B(s_0, s_0) = 1$ (when we fix a choice of s_0 in Lemma 5.3) and the operators f_v and $T_{\overline{v}}$ are adjoint with respect to B for all $v \in \mathfrak{p}$. Then the grading (5.15) is a B-orthogonal decomposition.

Taking inventory of our operators, we see that π^x, $x \in \mathfrak{k}$, and f_v, T_v, $v \in \mathfrak{p}$ are each graded operators on H of degrees 0,1, and -1 respectively. I.e., we have

$$\pi^x : H_t \to H_t, \qquad f_v : H_t \to H_{t+1}, \qquad T_v : H_t \to H_{t-1} \tag{6.17}$$

Now we can state

Theorem 6.3. *Suppose $\mathbf{N}^{\frac{1}{2}}$ is a half-form bundle on Y and $H = \Gamma(Y, \mathbf{N}^{\frac{1}{2}})$. Let*

$$\pi : \mathfrak{g} \to \operatorname{End}_{\mathfrak{k}-fin} H, \qquad z \mapsto \pi^z \tag{6.18}$$

be the complex linear map defined by (6.5), (6.6), (6.13) so that

$$\begin{aligned} \pi^x &= i\mathcal{Q}(\Phi^x) = \mathcal{L}_{\eta^x} && \text{if } x \in \mathfrak{k}, \\ \pi^v &= i\mathcal{Q}(\Phi^v) = if_v + iT_v && \text{if } v \in \mathfrak{p} \end{aligned} \tag{6.19}$$

Then, except in the one case where $G_{\mathbb{R}} = \widetilde{SL}(3,\mathbb{R})$, $r_0 = 1$, and $H_1 \simeq \mathbb{C}^4$, the map π is a complex Lie algebra homomorphism so that π is a representation of $\mathfrak{g}$ by global algebraic pseudo-differential operators on sections of $\mathbf{N}^{\frac{1}{2}}$.

Proof. As in [B-K4,§6], the problem reduces to proving the single bracket relation of operators on H

$$[\pi^e, \pi^{\overline{e}}] = \pi^h \tag{6.20}$$

because of [B-K4, Lem. 3.6]. Since the operators satisfy

$$[f_v, f_{v'}] = [T_v, T_{v'}] = 0 \tag{6.21}$$

for all $v, v' \in \mathfrak{p}$, we get $[\pi^e, \pi^{\overline{e}}] = [f_{\overline{e}}, T_e] - [f_e, T_{\overline{e}}]$. Thus (6.20) amounts to the relation

$$[f_{\overline{e}}, T_e] - [f_e, T_{\overline{e}}] = \mathcal{L}_{\eta^h} \tag{6.22}$$

We prove (6.22) in §7. We also show that in the $SL(3,\mathbb{R})$ case we omitted, π fails to be a Lie algebra homomorphism. ∎

For the rest of this section, we assume that we are in the situation of Theorem 6.3 with the one bad case excluded so that π is a Lie algebra homomorphism. Let

$$\widetilde{\pi} : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}_{\,\mathfrak{k}-fin} H \tag{6.23}$$

be the complex algebra homomorphism defined by π. Let $\mathcal{E}$ be the image of $\widetilde{\pi}$ and let $\mathcal{I} \subset \mathcal{U}(\mathfrak{g})$ be the kernel of $\widetilde{\pi}$. Then we have a natural complex algebra isomorphism

$$\mathcal{U}(\mathfrak{g})/\mathcal{I} \simeq \mathcal{E} \tag{6.24}$$

Let $S^{[p]}(\mathfrak{g})$ be the pth Cartan power of the adjoint representation of $\mathfrak{g}$.

Theorem 6.4. *The representation π of $\mathfrak{g}$ on H is irreducible. The algebra homomorphism $\widetilde{\pi}$ is surjective so that*

$$\mathcal{E} = \mathrm{End}_{\,\mathfrak{k}-fin} H \tag{6.25}$$

The algebra $\mathcal{E}$ has no zero-divisors. Thus the annihilator $\mathcal{I}$ of π is a completely prime primitive ideal in $\mathcal{U}(\mathfrak{g})$.

We have $\mathcal{E}^p/\mathcal{E}^{p-1} \simeq S^{[p]}(\mathfrak{g})$ as $\mathfrak{g}$-modules and so there is a multiplicity free $\mathfrak{g}$-module decomposition

$$\mathcal{E} \simeq \oplus_{p\in\mathbb{Z}_+} S^{[p]}(\mathfrak{g}) \tag{6.26}$$

The associated graded ideal $\operatorname{gr}\mathcal{I} \subset S(\mathfrak{g})$ *is the prime ideal defining the closure of* $O_{\min}$. *Thus the associated graded map to* $\widetilde{\pi}$ *gives a graded algebra isomorphism*

$$R(O_{\min}) \xrightarrow{\sim} \operatorname{gr}\mathcal{E} \tag{6.27}$$

Proof. The proofs in [B-K4,§5] of the corresponding results go through verbatim (by design) into this more general setting. The only change needed is that we replace the line "Let $s_0 = 1 \in H_{r_0}$" in [B-K4] by "Let $s_0 \in H_{r_0}$ be a non-zero highest weight vector for the $\mathfrak{k}$-action." ■

If $\mathfrak{g} \neq \mathfrak{sl}(n,\mathbb{C})$, then Joseph ([J2]) proved that $\mathcal{U}(\mathfrak{g})$ contains a unique completely prime primitive ideal with associated nilpotent orbit $O_{\min}$ (cf. §4). This is called the Joseph ideal. Thus we get

Corollary 6.5. *If* $\mathfrak{g} \neq \mathfrak{sl}(n,\mathbb{C})$ *then* $\mathcal{I}$ *is the Joseph ideal.*

We remark that this gives a new proof of Garfinkle's ([G]) result that the associated graded ideal of the Joseph ideal is prime.

We now fix a non-zero vacuum vector $s_0 \in H_{r_0}$ which is Q-semi-invariant, or equivalently, a highest weight vector for the $\mathfrak{k}$-action.

Theorem 6.6. H *admits a unique* $\mathfrak{g}_{\mathbb{R}}$*-invariant positive-definite Hermitian inner product* $\langle s|s'\rangle$ *such that* $\langle s_0|s_0\rangle = 1$. *This coincides with the inner product* B *found in Corollary* 6.2 *so that*

$$B(s,s') = \langle s|s'\rangle \tag{6.28}$$

Consequently, the representation of $\mathfrak{g}_{\mathbb{R}}$ *on* H *given by* π *integrates uniquely to give a unitary representation*

$$\pi_o : G_{\mathbb{R}} \to \text{Unit } \mathcal{H} \tag{6.29}$$

on the Hilbert space $\mathcal{H}$ *obtained by completing* H *with respect to* B. *Then* H *is the space of* $K_{\mathbb{R}}$*-finite vectors in* $\mathcal{H}$.

Proof. This follows by the proof of [B-K4, Th. 5.2], using the same modification described in Theorem 6.4. See, e.g., [W, §6.A.4] for a proof of Harish-Chandra's theorem that the $\mathfrak{g}_{\mathbb{R}}$-action on an admissible finitely generated $(\mathfrak{g},K)$-module S endowed with a $\mathfrak{g}_{\mathbb{R}}$-invariant positive-definite Hermitian inner product integrates to a unitary representation of $G_{\mathbb{R}}$ on the Hilbert space completion of S. We apply this with $H = S$. Indeed H is irreducible by Theorem 6.4 and so generated by any non-zero vector. Also H is admissible (i.e., all K-multiplicities are finite) since H is in fact multiplicity free by Lemma 5.3. ■

We will write $\langle s|s\rangle = \|s\|^2$. Theorems 6.4 and 6.6 give, in the language of §4, the representation theoretic result

Corollary 6.7. *π_o is a minimal unitary representation of $G_\mathbb{R}$ and H is its associated Harish-Chandra $(\mathfrak{g}, K)$-module.*

Theorem 6.8. *There exist positive real numbers a and b (depending on $G_\mathbb{R}$ and $\mathbf{N}^{\frac{1}{2}}$) such that*

$$\left\| \frac{f_0^n s_0}{n!} \right\|^2 = \frac{(a)_n (b)_n}{n!(r_0+1)_n} \tag{6.30}$$

for all $n \in \mathbb{Z}_+$. Moreover a and b are unique up to ordering and satisfy

$$a + b = r_0 + 1 + X_0 \tag{6.31}$$

where X_0 is the eigenvalue of $\mathcal{L}_{\eta^h}$ on s_0. We compute a and b below in Table 6.9.

The values $\|f_0^n s_0\|^2$ and $K_\mathbb{R}$-invariance uniquely determine the inner product B on H because of the ladder decomposition (5.15).

Here we are using the hypergeometric function notation $(a)_n = a(a+1)\cdots(a+n-1)$.

Proof. The adjoint of multiplication by $f_0 = f_e$ is $T_{\overline{e}}$ by Corollary 6.2. We find $T_{\overline{e}}(f_0^k s_0) = \gamma_k f_0^{k-1} s_0$ for $k \in \mathbb{Z}_+$ where γ_k is a scalar and $\gamma_0 = 0$. This follows by E'-degree and weight as in [B-K4, proof of Th. 5.2]. Indeed, $T_{\overline{e}}(f_0^k s_0)$ lies in H_{r_0+k-1} and has $\mathfrak{h}$-weight $\nu + (k-1)\psi$. But by Lemma 5.3, $f_0^{k-1} s_0$ is a highest weight vector of weight $\nu + (k-1)\psi$ in the irreducible $\mathfrak{k}$-representation H_{r_0+k-1}. Thus $T_{\overline{e}}(f_0^k s_0)$ is a multiple of $f_0^{k-1} s_0$. We now find

$$\|f_0^n s_0\|^2 = \langle s_0 \,|\, T_{\overline{e}}^n (f_0^n s_0)\rangle = \gamma_1 \cdots \gamma_n \langle s_0 \,|\, s_0\rangle \tag{6.32}$$

We evaluate the RHS of (6.32) in §7 below and obtain (6.30) and (6.31). The final statement that these values determine B is immediate from Lemma 5.3 – in particular the ladder decomposition of H is multiplicity-free. ■

Table 6.9.

Case	$G_\mathbb{R}$	s_0^2	$V_\nu \simeq H_{r_0}$	r_0	a	b
(i)	$E_{6(6)}$	Λ_0	$\mathbb{C}$	$1+\frac{3}{2}d=\frac{5}{2}$	$1+\frac{1}{2}d=\frac{3}{2}$	$1+d=2$
(ii)	$E_{7(7)}$	Λ_0	$\mathbb{C}$	$1+\frac{3}{2}d=4$	$1+\frac{1}{2}d=2$	$1+d=3$
(iii)	$E_{8(8)}$	Λ_0	$\mathbb{C}$	$1+\frac{3}{2}d=7$	$1+\frac{1}{2}d=3$	$1+d=5$
(iv)	$F_{4(4)}$	Λ_0	$\mathbb{C}\otimes S^1\mathbb{C}^2$	$1+d=2$	$1+\frac{1}{2}d=\frac{3}{2}$	$1+d=2$
(v)	$E_{6(2)}$	Λ_0	$\mathbb{C}\otimes S^2\mathbb{C}^2$	$1+d=3$	$1+\frac{1}{2}d=2$	$1+d=3$
(vi)	$E_{7(-5)}$	Λ_0	$\mathbb{C}\otimes S^4\mathbb{C}^2$	$1+d=5$	$1+\frac{1}{2}d=3$	$1+d=5$
(vii)	$E_{8(-24)}$	Λ_0	$\mathbb{C}\otimes S^8\mathbb{C}^2$	$1+d=9$	$1+\frac{1}{2}d=5$	$1+d=9$
(viii)	$G_{2(2)}$	Λ_0	$S^2\mathbb{C}^2\otimes\mathbb{C}$	1	$1+\frac{1}{2}d=\frac{4}{3}$	$1+d=\frac{5}{3}$
(ix)	$\widetilde{SO}(p,q)$ $3\le p\le q$, $p+q$ is even	Λ_0	$S_o^{(q-p)/2}\mathbb{C}^p\otimes\mathbb{C}$	$\frac{q-2}{2}$	$\frac{q-2}{2}$	$\frac{q-p+2}{2}$

(x)	$\widetilde{SO}(3,q)$ $4 \le q$, q is even	Λ_0	$S^{q-3}\mathbb{C}^2 \otimes \mathbb{C}$	$\frac{q-2}{2}$	$\frac{q-2}{2}$	$\frac{q-1}{2}$
(xi)	$\widetilde{SO}(3,3)$	$f_0\Lambda_0$	$\mathbb{C}^2 \otimes \mathbb{C}^2$	1	$\frac{3}{2}$	$\frac{3}{2}$
(xii)	$\widetilde{SO}(p,q)$ $4 \le p \le q$, $p+q$ is odd	none	$*$	$*$	$*$	$*$
(xiii)	$\widetilde{SL}(n,\mathbb{R})$ $4 \le n$, n is even	Λ_0	$\mathbb{C}$	$\frac{n-2}{4}$	$\frac{1}{2}$	$\frac{n}{4}$
(xiv)	$\widetilde{SL}(n,\mathbb{R})$ $4 \le n$, n is even	$f_0\Lambda_0$	$\mathbb{C}^n$	$\frac{n}{4}$	$\frac{3}{2}$	$\frac{n+2}{4}$
(xv)	$\widetilde{SL}(n,\mathbb{R})$ $5 \le n$, n is odd	none	$*$	$*$	$*$	$*$
(xvi)	$\widetilde{SL}(3,\mathbb{R})$	Λ_0	$\mathbb{C}^2$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{4}$
(xvii)	$\widetilde{SL}(3,\mathbb{R})$	$f_0\Lambda_0$	$S^3\mathbb{C}^2$	1	$*$	$*$

In Table 6.9, the symbol $*$ means that there is no entry because some aspect of the construction has failed. In Cases (xii) and (xv) there is no half-form bundle, while in Case (xvii) the operators π^z fail to satisfy the bracket relations of $\mathfrak{sl}(3,\mathbb{R})$.

The final result we present here about our minimal representations is the computation of a matrix coefficient on the one parameter subgroup generated by $x = e + \overline{e} \in \mathfrak{p}_{\mathbb{R}}$. Indeed, the same arguments used in [B-K4, Th. 6.6] go though in this setting to give

Theorem 6.10. *We have, for $t \in \mathbb{R}$,*

$$\langle (\exp\ tx) \cdot s_0 | s_0 \rangle = \ {}_2F_1(a, b; 1 + r_0; -\sinh^2 t) \tag{6.33}$$

§7. Differential Operators on Half-forms and the Generalized Capelli Identity.

The purpose of this section is to complete the proofs of the results in §6, i.e., to prove Theorems 6.3 and 6.8. We already reduced Theorem 6.3 to the operator relation (6.22) on H.

To begin the proof of (6.22), we observe that the two operators $[f_{\overline{e}}, T_e] - [f_e, T_{\overline{e}}]$ and $\mathcal{L}_{\eta^h}$ appearing on the LHS and the RHS of (6.22) are both K_0-invariant. The first idea of the proof is to exploit this observation. Since K_0 is reductive, it follows that H is completely reducible as K_0-representation. So we can fix a direct sum decomposition $H = \oplus_\alpha H_\alpha$ where each subspace H_α carries an irreducible K_0-representation. Now to prove the operator relation (6.22) holds on H_α it suffices to prove that (6.22) holds on just one non-zero section s^α in H_α.

There is a natural method to pick a section s^α from H_α, unique up to scaling. This uses structure of H_α as a $\mathfrak{k}_0$-representation. The method is to pick s^α to be a lowest weight vector for $\mathfrak{k}_0$. Here we appeal again to the Cartan-Weyl theory recalled in §5 (where we applied it

to H considered as a $\mathfrak{k}$-representation). This time we use lowest weights rather than highest weights just for convenience.

To get the notion of lowest weight for $\mathfrak{k}_0$, we take the triangular decomposition

$$\mathfrak{k}_0 = \mathfrak{h} \oplus \mathfrak{m}_0 \oplus \mathfrak{m}_0^- \tag{7.1}$$

induced by (5.7). Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{k}_0$ and $\mathfrak{b}_0 = \mathfrak{h} \oplus \mathfrak{m}_0$ is a Borel subalgebra. So now a *lowest weight vector* in H_α for $\mathfrak{k}_0$ is a vector in the 1-dimensional space $H_\alpha^{\mathfrak{m}_0^-}$. Then

$$H^{\mathfrak{m}_0^-} = \bigoplus_\alpha H_\alpha^{\mathfrak{m}_0^-} \tag{7.2}$$

is the space of (i.e., spanned by) all lowest weight vectors in H for $\mathfrak{k}_0$.

So proving (6.22) reduces to verifying it on each lowest weight vector $s^\alpha \in H_\alpha^{\mathfrak{m}_0^-}$. By K_0-invariance and Schur's Lemma , the vectors $([f_{\overline{e}}, T_e] - [f_e, T_{\overline{e}}])(s^\alpha)$ and $\mathcal{L}_{\eta^h}(s^\alpha)$ again lie in $H_\alpha^{\mathfrak{m}_0^-}$. Consequently, for $s = s^\alpha$ we have

$$\begin{aligned} \mathcal{L}_{\eta^h}(s) =& Xs \\ ([f_{\overline{e}}, T_e] - [f_e, T_{\overline{e}}])(s) =& Rs \end{aligned} \tag{7.3}$$

where X and R are scalars depending on α. So proving (6.22) reduces to showing that, for each α, the scalars X and R coincide.

The second, more serious, idea is to work out the first idea using the Jordan algebra structure on $\mathfrak{k}_{-1}$ from §4. It turns out that the Jordan structure gives us (i) a nice way to write down a basis of $H^{\mathfrak{m}_0^-}$ and (ii) a means to compute R in (7.3) in the form of the generalized Capelli Identity of Kostant and Sahi ([K-S]). The computation of X follows easily from (i). Then, with everything computed, we see manifestly that $R = X$.

The rest of this section is devoted to working this out. To start off, we embed H in a larger space $H^\sharp$ which is easier to work with. We choose the natural embedding

$$H \subset H^\sharp = \Gamma(\widetilde{Y^o}, \mathbf{N}^{\frac{1}{2}}) \tag{7.4}$$

We constructed the 2-fold covering $\widetilde{Y^o}$ of Y^o in (5.27) and then the pullback bundle (again called) $\mathbf{N}^{\frac{1}{2}}$. Clearly H sits inside $H^\sharp$ as the space of sections which descend to Y^o (i.e., are $(\mathbb{Z}/2\mathbb{Z})$-invariant) and then extend to all of Y.

In Corollary 5.10 we got an nice description in (5.28) of $H^\sharp$. Since f_0 and $\sqrt{\Lambda}$ are K_0-semi-invariant, we might as well rewrite (5.28) as

$$H^\sharp = \left(\mathbb{C}[f_1, \ldots, f_m] \otimes \mathbb{C}[f_0^{\frac{1}{2}}, f_0^{-\frac{1}{2}}]\right) \sqrt{\Lambda} \tag{7.5}$$

This makes it clear that a decomposition of the polynomial algebra $\mathbb{C}[f_1, \ldots, f_m]$ into irreducible K_0-representations will produce a decomposition $H^\sharp = \oplus_\alpha H_\alpha^\sharp$ into irreducible K_0-

representations. In particular we have

$$(H^\sharp)^{\mathfrak{m}_0^-} = \left(\mathbb{C}[f_1,\dots,f_m]^{\mathfrak{m}_0^-} \otimes \mathbb{C}[f_0^{\frac{1}{2}}, f_0^{-\frac{1}{2}}]\right)\sqrt{\Lambda} \tag{7.6}$$

We will deal with the problem of locating $H^{\mathfrak{m}_0^-}$ inside $(H^\sharp)^{\mathfrak{m}_0^-}$ when the time comes.

Now we can bring in the Jordan algebra $\mathfrak{k}_{-1}$. We have a vector space isomorphism $\mathfrak{k}_{-1} \to \mathfrak{p}_1$, $y \mapsto [y, e]$; cf. Lemma 3.4. This induces a graded complex algebra isomorphism

$$S(\mathfrak{k}_{-1}) \to \mathbb{C}[f_1,\dots,f_m], \qquad g \mapsto \hat{g} \tag{7.7}$$

defined in degree 1 by $\hat{y} = f_{[y,e]}$ for $y \in \mathfrak{k}_{-1}$ where f_v was defined in (3.1). This isomorphism has weight χ^p in degree p under the action of K_0. Hence (7.7) is K_0'-equivariant and so gives by restriction a graded complex algebra isomorphism

$$S(\mathfrak{k}_{-1})^{\mathfrak{m}_0^-} \to \mathbb{C}[f_1,\dots,f_m]^{\mathfrak{m}_0^-}, \qquad g \mapsto \hat{g} \tag{7.8}$$

Recall from (4.12) that q_n is the degree of the Jordan algebra $\mathfrak{j}_{[n]}$. We put $c_1 = 0$ and

$$c_n = q_1 + \cdots + q_{n-1}, \qquad n = 2,\dots,\ell \tag{7.9}$$

We have the well-known result (see [B-K3, Theorem 3.4 and Corollary 3.6])

Lemma 7.1. *The natural representation of K_0 on $S(\mathfrak{k}_{-1})$ is completely reducible and multiplicity free.*

The ring of lowest weight vectors in $S(\mathfrak{k}_{-1})$ for the K_0-action is a polynomial algebra in q independent graded generators so that

$$S(\mathfrak{k}_{-1})^{\mathfrak{m}_0^-} = \mathbb{C}[N_1,\dots,N_q] \tag{7.10}$$

The polynomials $N_1,\dots,N_q$ are uniquely determined by the conditions (i) *for $1 \le j \le q_n$, we have $N_{c_n+j} \in S^j(\mathfrak{j}_{[n]})$ and* (ii) *$N_1(e_{\mathfrak{k}}) = \cdots = N_q(e_{\mathfrak{k}}) = 1$.*

Now combining the work we have done thus far in this section we obtain

Proposition 7.2. *The ring of lowest weight vectors in $\mathbb{C}[f_1,\dots,f_m]$ for the K_0-action is a polynomial algebra in the q independent graded generators $\hat{N}_1,\dots,\hat{N}_q$ so that*

$$\mathbb{C}[f_1,\dots,f_m]^{\mathfrak{m}_0^-} = \mathbb{C}[\hat{N}_1,\dots,\hat{N}_q] \tag{7.11}$$

Then $\deg \hat{N}_{c_n+j} = j$ for $1 \le j \le q_n$.

Suppose $\mathbf{N}^{\frac{1}{2}}$ is a half-form bundle on Y and $H = \Gamma(Y, \mathbf{N}^{\frac{1}{2}})$. Then the natural K_0-action on H is completely reducible and has a basis of lowest weight vectors of the form

$$s = f_0^p \hat{N}_1^{t_1} \cdots \hat{N}_q^{t_q} \sqrt{\Lambda} \tag{7.12}$$

where $p \in \frac{1}{2}\mathbb{Z}$ and $t_1, \ldots, t_q \in \mathbb{Z}_+$. Here we regard s as a section in the larger space $H^\sharp$. The section s in (7.12) *determines $p, t_1, \ldots, t_q$ uniquely.*

The K_0-representation on each E'-eigenspace H_{r_0+n} is multiplicity free.

If we range over all tuples $p, t_1, \ldots, t_q$ in (7.12), then we obtain a basis of $(H^\sharp)^{\mathfrak{m}_0^-}$. We could explain how to decide when the corresponding section lies in H, but we omit this as it is not necessary in our work below. (Only the partial answer we give later is needed.)

Now that we have obtained a nice basis of $H^{\mathfrak{m}_0^-}$ as promised, we need to start computing the eigenvalues of our various operators on the section s in (7.12). We set $\mathbf{t} = (t_1, \ldots, t_q)$. We put

$$N^{\mathbf{t}} = N_1^{t_1} \cdots N_q^{t_q} \qquad \text{and} \qquad z = \deg N^{\mathbf{t}} = \sum_{i=1}^{q} t_i \deg N_i \tag{7.13}$$

Recall from (3.14) that $m = \dim_{\mathbb{C}} Y - 1$.

Lemma 7.3. *Suppose $s \in H$ is of the form* (7.12) *Then*

$$\begin{aligned} E's =& rs \qquad \textit{where} \qquad r = p + z + \frac{m+1}{2} \\ \mathcal{L}_{\eta^h} s =& Xs \qquad \textit{where} \qquad X = 2p + z + \frac{m+2}{2} \end{aligned} \tag{7.14}$$

Proof. We have

$$E'(s) = (E f_0^p \hat{N}^{\mathbf{t}})\sqrt{\Lambda} + f_0^p \hat{N}^{\mathbf{t}}(E'\sqrt{\Lambda}) = (p+z)s + \frac{m+1}{2}s \tag{7.15}$$

using (7.12) and the formulas (3.19) and (5.17) for E and Λ in terms of our local coordinates. Next

$$\eta^h = 2f_0 \frac{\partial}{\partial f_0} + \sum_{i=1}^{m} f_i \frac{\partial}{\partial f_i} \tag{7.16}$$

Then computing the same way as in (7.15) we find

$$\mathcal{L}_{\eta^h}(s) = (\eta^h f_0^p \hat{N}^{\mathbf{t}})\sqrt{\Lambda} + f_0^p \hat{N}^{\mathbf{t}}(\mathcal{L}_{\eta^h}\sqrt{\Lambda}) = (2p+z)s + \frac{m+2}{2}s \tag{7.17}$$

■

Now we have come to the tricky part, computing the eigenvalue R of $[f_{\overline{e}}, T_e] - [f_e, T_{\overline{e}}]$ on each section s from (7.12). Expanding out we get

$$\begin{aligned} &[f_{\overline{e}}, T_e] - [f_e, T_{\overline{e}}] = f_{\overline{e}} T_e - T_e f_{\overline{e}} - f_e T_{\overline{e}} + T_{\overline{e}} f_e \\ &= \frac{1}{(E'-1)E'} f_{\overline{e}} D_e - \frac{1}{E'(E'+1)} D_e f_{\overline{e}} - \frac{1}{(E'-1)E'} f_e D_{\overline{e}} + \frac{1}{E'(E'+1)} D_{\overline{e}} f_e \end{aligned} \tag{7.18}$$

We will use the generalized Capelli Identity of Kostant and Sahi ([K-S]) to compute the eigenvalues of $f_{\overline{e}}D_e$, $D_e f_{\overline{e}}$, $f_e D_{\overline{e}}$, $D_{\overline{e}} f_e$, , and on s. The idea is to transform this computation into a computation on $S(\mathfrak{k}_{-1})$. This works because we will write everything in terms of our local coordinates $f_0, f_1, \ldots, f_m$ and use (7.7).

The first thing we will compute is $f_{\overline{e}}D_e(s)$. We already have the expression (6.11) for D_e in terms of our local coordinates. So we need the expression for the function $f_{\overline{e}}$. In analogy to (4.13) we set

$$N = N_{q_1}^{w_1} N_{q_1+q_2}^{w_2} \cdots N_{q_1+\cdots+q_\ell}^{w_\ell} \tag{7.19}$$

Complex conjugation on $\mathfrak{g}$ preserves $O_{\min}$ and $\mathfrak{p}$ and so preserves Y by (3.7). This induces naturally a complex conjugation map $f \mapsto \overline{f}$ on $R(Y)$. This is then a $\mathbb{C}$-anti-linear real algebra involution.

Proposition 7.4. *The unique expression for $f_{\overline{e}}$ in terms of our local coordinates $f_0, f_1, \ldots, f_m$ on Y is*

$$f_{\overline{e}} = \overline{f}_0 = \frac{\hat{N}}{f_0^3} \tag{7.20}$$

Proof. This is proven in the same way as in [B-K4, Prop. 4.3]. ■

To state and prove our computation of $f_{\overline{e}}D_e(s)$, we need to encode the monomial $N^{\mathbf{t}}$ into a new q-vector, namely the *multi-degree* $\mathbf{deg}(N^{\mathbf{t}})$. We define this by

$$\mathbf{deg}(N_{c_n+j}) = (0, \ldots, 0, 2, \ldots, 2, 0, \ldots, 0), \qquad 1 \le j \le q_n \tag{7.21}$$

where there are c_n zeroes follows by j twos followed by zeroes, and

$$\mathbf{deg}(N^{\mathbf{t}}) = \sum_{i=1}^{q} t_i \mathbf{deg}(N_i) \tag{7.22}$$

where addition of vectors is component-wise. If $\mu = (\mu_1, \ldots, \mu_q) = \mathbf{deg}(N^{\mathbf{t}})$ then

$$\mu_1 + \cdots + \mu_q = 2 \deg N^{\mathbf{t}} = 2z \tag{7.23}$$

Let δ be the q-vector such that $\delta_{c_n+j} = d_n(q_n - j)$ for $1 \le j \le q_n$ where we recall the root multiplicity numbers d_n from the proof of Proposition 5.7. So

$$\begin{aligned} \delta &= (\delta_1, \ldots, \delta_q) \\ &= (d_1(q_1-1), \ldots, d_1, 0, d_2(q_2-1), \ldots, d_2, 0, \ldots d_\ell(q_\ell-1), \ldots, d_\ell, 0) \end{aligned} \tag{7.24}$$

Finally we define the q-vector

$$\mathbf{v} = (v_1, \ldots, v_q) = (w_1, \ldots, w_1, \ldots, w_\ell, \ldots, w_\ell) \tag{7.25}$$

where each w_n occurs q_n times.

From now on, we assume that $s \in H$ is of the form $s = f_0^p \hat{N}^{\mathbf{t}} \sqrt{\Lambda}$ of (7.12) and $\mu = \mathbf{deg}(N^{\mathbf{t}})$.

Proposition 7.5. *We have*

$$f_{\overline{e}} D_e(s) = s \prod_{i=1}^{q} \prod_{j=0}^{v_i - 1} C_{i,j}(\mu) \tag{7.26}$$

where $C_{i,j}(\mu)$ is the Capelli multiplier given by

$$C_{i,j}(\mu) = \frac{\mu_i + \delta_i - 2j}{2v_i} \tag{7.27}$$

Proof. First (6.11) and (7.20) give

$$f_{\overline{e}} D_e = \hat{N} P(\mathcal{L}_{\partial_1}, \ldots, \mathcal{L}_{\partial_m}) \tag{7.28}$$

where $\partial_k = \frac{\partial}{\partial f_k}$ for $k = 1, \ldots, m$. To compute $f_{\overline{e}} D_e(s)$ we first compute $\mathcal{L}_{\partial_k}(s)$. But $\partial_k(f_0) = 0$ and also taking the Lie derivative of (5.17) we get $\mathcal{L}_{\partial_k}(\sqrt{\Lambda}) = 0$. So we get

$$\mathcal{L}_{\partial_k}(s) = \partial_k(f_0^p \hat{N}^{\mathbf{t}}) \sqrt{\Lambda} + f_0^p \hat{N}^{\mathbf{t}} (\mathcal{L}_{\partial_k} \sqrt{\Lambda}) = f_0^p (\partial_k \hat{N}^{\mathbf{t}}) \sqrt{\Lambda} \tag{7.29}$$

Then (7.28) gives

$$f_{\overline{e}} D_e(s) = f_0^p \hat{N} \left(P(\partial_1, \ldots, \partial_m)(\hat{N}^{\mathbf{t}}) \right) \sqrt{\Lambda} \tag{7.30}$$

Next we introduce the graded complex algebra isomorphism

$$S(\mathfrak{k}_1) \to \mathbb{C}\left[\partial_1, \ldots, \partial_m\right], \qquad B \mapsto \partial_B \tag{7.31}$$

defined in degree 1 by $\partial_x = f_0^{-1} \eta^x$, $x \in \mathfrak{k}_1$, so that, by (3.19), $\partial_{x_k} = \frac{\partial}{\partial f_k}$. Then we can rewrite (7.30) as

$$f_{\overline{e}} D_e(s) = f_0^p \hat{N} \left(\partial_P(\hat{N}^{\mathbf{t}}) \right) \sqrt{\Lambda} \tag{7.32}$$

It follows easily, as in [B-K4, (4.4.4)], that

$$\hat{N} \partial_P(\hat{N}^{\mathbf{t}}) = \widehat{N \partial_P(N^{\mathbf{t}})} \tag{7.33}$$

Indeed, it suffices to check that $\partial_x \hat{y} = \partial_x y$ where $x \in \mathfrak{k}_1$ and $y \in \mathfrak{k}_{-1}$. We find $\partial_x \hat{y} = f_e^{-1} f_{\psi([x,y])e} = \psi([x,y]) = (x,y)_{\mathfrak{g}} = \partial_x y$.

Now we can compute $N\partial_P(N^{\mathbf{t}})$ using the generalized Capelli Identity of Kostant and Sahi [K-S]. This is similar to [B-K4, proof of Th. 4.4], but we are in a more general situation here

where the Jordan algebra $\mathfrak{k}_{-1}$ is not necessarily simple and also the multiplicities $w_1, \ldots, w_\ell$ may be non-trivial. The point is that $N\partial_P(N^{\mathbf{t}})$ breaks into a product with one factor for each simple component $\mathfrak{j}_{[n]}$ of $\mathfrak{k}_{-1}$. The nth factor is

$$N_{c_n+q_n}^{w_n} \partial_{P_{[n]}^{w_n}} \left(N_{c_n+1}^{t_{c_n+1}} \cdots N_{c_n+q_n}^{t_{c_n+q_n}} \right) \tag{7.34}$$

The Capelli Identity for $\mathfrak{j}_n$ says that the operator $N_{c_n+q_n}^{w_n} \partial_{P_{[n]}^{w_n}}$ acts on it argument in (7.34) by a scalar and computes that scalar.

Putting all the factors together we obtain

$$N\partial_P(N^{\mathbf{t}}) = N^{\mathbf{t}} \prod_{i=1}^{q} \prod_{j=0}^{v_i-1} C_{i,j}(\mu) \tag{7.35}$$

which then gives (7.26) because of (7.32) and (7.33).

There is one subtle point here: the appearance of the factor v_i^{-1} in $C_{i,j}(\mu)$. The corresponding Capelli multiplier from [K-S] is just $\frac{1}{2}(\mu_i + \delta_i - 2j)$. However the factor v_i^{-1} arises because of the way we have paired $\mathfrak{k}_{-1}$ with $\mathfrak{k}_1$. Let $e_{\mathfrak{k}} = e_{\mathfrak{k}}^1 + \cdots + e_{\mathfrak{k}}^\ell$ be the decomposition of $e_{\mathfrak{k}}$ corresponding to (4.11) so that $e_{\mathfrak{k}}^n$ is the Jordan identity element in $\mathfrak{j}_{[n]}$; similarly we get $\overline{e}_{\mathfrak{k}} = \overline{e}_{\mathfrak{k}}^1 + \cdots + \overline{e}_{\mathfrak{k}}^\ell$. Then we easily find

$$(e_{\mathfrak{k}}, \overline{e}_{\mathfrak{k}})_{\mathfrak{g}} = \sum_{n=1}^{\ell} (e_{\mathfrak{k}}^n, \overline{e}_{\mathfrak{k}}^n)_{\mathfrak{g}} \qquad \text{and} \qquad (e_{\mathfrak{k}}^n, \overline{e}_{\mathfrak{k}}^n)_{\mathfrak{g}} = q_n w_n \tag{7.36}$$

This fits with (4.14) since $(e_{\mathfrak{k}}, \overline{e}_{\mathfrak{k}})_{\mathfrak{g}} = 4$ just as in [B-K4, proof of Theorem 4.4.) So we get $\partial_{e_{\mathfrak{k}}^n}(\overline{e}_{\mathfrak{k}}^n) = q_n w_n$. However the normalization from [K-S] is that $\partial_{e_{\mathfrak{k}}^n}(\overline{e}_{\mathfrak{k}}^n) = q_n$. The ratio w_n then appears in the denominator of our Capelli multiplier. ■

Let Ξ be the set of ordered pairs (i,j) occurring in (7.26). The cardinality of Ξ is $q_1 w_1 + \ldots + q_n w_n = 4$ by (4.14).

Corollary 7.6. *We have*

$$D_e f_{\overline{e}}(s) = s \prod_{(i,j)\in\Xi} \left(C_{i,j}(\mu) + 1 \right) \tag{7.37}$$

Proof. As in the last proof we find, cf. (7.28),

$$D_e f_{\overline{e}} = P(\mathcal{L}_{\partial_1}, \ldots, \mathcal{L}_{\partial_m}) \hat{N} \tag{7.38}$$

and so, as in (7.32) and (7.33),

$$D_e f_{\overline{e}}(s) = f_0^p(\widehat{\partial_P(N N^{\mathbf{t}})}) \sqrt{\Lambda} \tag{7.39}$$

Then we find

$$\partial_P(NN^{\mathbf{t}}) = \prod_{(i,j)\in\Xi} C_{i,j}(\mu') \tag{7.40}$$

where $\mu' = \mathbf{deg}(NN^{\mathbf{t}})$. But $\mu' = 2\mathbf{v} + \mu$ and so $C_{i,j}(\mu') = C_{i,j}(\mu) + 1$. Now we get (7.37). ■

Next we want compute $f_e D_{\overline{e}}(s)$. We solve this as in [B-K4] by introducing, in the next result, an involution θ of H. We construct this using the group element

$$\theta_o = \exp \frac{\pi}{2}(e_{\mathfrak{k}} - \overline{e}_{\mathfrak{k}}) \in K \tag{7.41}$$

where $\exp : \mathfrak{k} \to K$ is the exponential map. The same arguments used in [B-K4, Lem. 4.6, Prop. 4.6 and Th. 4.7] give

Lemma 7.7. *The action of θ_o on $\mathfrak{p}$ preserves Y and defines a graded complex algebra involution θ of $R(Y)$ which commutes with complex conjugation. We have $f_0^\theta = \pm\overline{f}_0$.*

The natural action of θ_o on the K-homogeneous half-form bundle $\mathbf{N}^{\frac{1}{2}}$ over Y defines a complex linear involution $\theta : H \to H$. Then $\theta : H \to H$ preserves the grading (5.15) *and is compatible with the $R(Y)$-module structure so that $(fs)^\theta = f^\theta s^\theta$.*

θ permutes the simple $\mathfrak{k}_0$-submodules in H and moves lowest weight vectors to highest weight vectors. For any $s \in H$ we have

$$f_e D_{\overline{e}}(s) = (f_{\overline{e}} D_e(s^\theta))^\theta \tag{7.42}$$

We can now prove

Proposition 7.8. *We have*

$$f_e D_{\overline{e}}(s) = s \prod_{(i,j)\in\Xi} \big(r - 1 - C_{i,j}(\mu)\big) \tag{7.43}$$

Proof. Lemma 7.7 reduces the problem to computing $f_{\overline{e}} D_e(s^\theta)$. Now s is a lowest weight vector in some simple $\mathfrak{k}_0$-submodule F in H, and so Lemma 7.7 implies that s^θ is a highest weight vector in the simple $\mathfrak{k}_0$-submodule F^θ in H. Then there is a lowest weight vector $s^* \in F^\theta$ (unique up to scaling). We can write $s^* = f_0^a \hat{u}\sqrt{\Lambda}$ where $u \in (S^b(\mathfrak{k}_{-1}))^{\mathfrak{m}_0^-}$. Let $\mu^* = \mathbf{deg}(u)$. Then

$$f_{\overline{e}} D_e(s^\theta) = s^\theta \prod_{(i,j)\in\Xi} C_{i,j}(\mu^*) \tag{7.44}$$

We claim that there is an involution $(i,j) \mapsto (i^*, j^*)$ on the set Ξ such that

$$C_{i^*,j^*}(\mu^*) = r - 1 - C_{i,j}(\mu) \tag{7.45}$$

This, because of Lemma 7.7, gives (7.43).

The construction of the involution requires several calculations. To carry these out, we bring to the forefront the theory of weights associated to our triple $(\mathfrak{k}, \mathfrak{k}_0, \mathfrak{k}^{\mathfrak{l}})$ from §4. Indeed, our choice of $(\mathfrak{h}, \mathfrak{b})$ in §5 was compatible with complex conjugation and the complex Cartan decomposition $\mathfrak{k}_0 = \mathfrak{k}^{\mathfrak{l}} \oplus \mathfrak{r}$ so that we have a complex conjugation stable splitting $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ where $\mathfrak{a} \subset \mathfrak{r}$ is a maximal abelian subalgebra and $\mathfrak{t} \subset \mathfrak{k}^{\mathfrak{l}}$.

Now $\dim_{\mathbb{C}} \mathfrak{a} = q$ and $\mathfrak{a}^*$ has a unique basis $\varepsilon_1, \dots, \varepsilon_q$ such that the $\mathfrak{a}$-weight of N_{c_n+j}, where $1 \leq j \leq q_n$, is $-2(\varepsilon_{c_n+1} + \dots + \varepsilon_{c_n+j})$. The weights ε_i are pure imaginary and the action of θ_o gives the complex involution θ of $\mathfrak{k}_0$ with fixed algebra $\mathfrak{k}^{\mathfrak{l}}$ and (-1)-eigenspace $\mathfrak{r}$. In particular, θ_o acts as -1 on $\mathfrak{a}$. It follows that

$$\sigma = \mathfrak{a}\text{-weight of } s \quad \Rightarrow \quad -\sigma = \mathfrak{a}\text{-weight of } s^\theta \tag{7.46}$$

Let $\nu \mapsto \nu^\tau$ be the involution of $\mathfrak{a}^*$ which exchanges the highest weight of a simple $\mathfrak{k}_0$-submodule in $S(\mathfrak{k}_{-1})$ with the lowest weight. Then

$$-\sigma^\tau = \mathfrak{a}\text{-weight of } s^* \tag{7.47}$$

In terms of our basis of $\mathfrak{a}^*$ we have $\varepsilon^\tau_{c_n+i} = \varepsilon_{c_n+q_n-i+1}$ for $1 \leq i \leq q_n$. The $\mathfrak{a}$-weight of $f_0 = f_e$ is $\psi|_{\mathfrak{a}}$, which we again call ψ. From now on, we identify a q-vector $\nu = (\nu_1, \dots, \nu_q)$ with the $\mathfrak{a}$-weight $\nu = \sum_{i=1}^q \nu_i \varepsilon_i$.

We now define our involution on Ξ by

$$\varepsilon_{i^*} = \varepsilon_i^\tau \qquad \text{and} \qquad j^* = v_i - j - 1 \tag{7.48}$$

(This automorphism of Ξ is can happen to be the identity, so by involution we mean just that the order divides 2.) Then $\mu_i^\tau = \mu_{i^*}$.

Let us put $g = N^{\mathbf{t}}$ so that $s = f_0^p \hat{g}\sqrt{\Lambda}$, $\mu = \mathbf{deg}(g)$ and $g \in (S^z(\mathfrak{k}_1))^{\mathfrak{m}_0^-}$. Then the $\mathfrak{a}$-weight of $\hat{g}$ is $z\psi - \mu$. Also

$$\lambda = \mathfrak{a}\text{-weight of } \sqrt{\Lambda} = \frac{m+1}{2}\psi - \frac{\kappa}{2} \tag{7.49}$$

where $-\kappa$ is the weight of $\mathfrak{a}$ on $\wedge^m \mathfrak{k}_{-1}$. The sum of the $\mathfrak{a}$-weights of f_0^p, $\hat{g}$ and $\sqrt{\Lambda}$ is

$$\sigma = (p+z)\psi - \mu + \lambda = r\psi - \mu - \frac{\kappa}{2} \tag{7.50}$$

Now we can compute μ^*. The sections $s = f_0^p \hat{g}\sqrt{\Lambda}$ and $s^* = f_0^a \hat{u}\sqrt{\Lambda}$ have the same eigenvalues under E' and $\mathcal{L}_{\eta^h}$ which means that $p + z = a + b$ and $2p + z = 2a + b$. Hence $p = a$ and $z = b$. So the $\mathfrak{a}$-weight of $\hat{u}$ is $z\psi - \mu^*$ and we get

$$-\sigma^\tau = \mathfrak{a}\text{-weight of } s^* = (p+z)\psi - \mu^* + \lambda \tag{7.51}$$

Applying τ to (7.49) and subtracting (7.50) we get

$$\mu^* = 2\sigma^\tau + \mu^\tau = 2r\psi - \mu^\tau - \kappa \tag{7.52}$$

To obtain a proof of (7.45), we write out (7.52) in terms of components. The $\mathfrak{a}$-weight ψ has the same components as the vector $\mathbf{v}$ in (7.25). A key observation is that the components of κ are $\kappa_i = 2 + \delta_i + \delta_i^\tau$. To see this, we start from the fact that κ is the sum of the $\mathfrak{a}$-weights $\kappa^{[n]}$ of the top exterior powers of the spaces $\mathfrak{j}_{[n]}$. The weights of $\mathfrak{a}$ on $\mathfrak{j}_{[n]}$ are precisely the weights $2\varepsilon_i$ and $\varepsilon_i + \varepsilon_j$ where $c_n + 1 \le i < j \le c_n + q_n$. So $\kappa^{[n]} = 2\sum_i \varepsilon_i + d_n \sum_{i<j}(\varepsilon_i + \varepsilon_j)$ and this gives our formula for κ_i. Now (7.52) gives

$$\mu_i^* + \delta_i = 2rv_i - 2 - (\mu_i^\tau + \delta_i^\tau) \tag{7.53}$$

Now subtracting $2j$ from both sides of (7.53) and using (7.48) we get

$$\mu_i^* + \delta_i - 2j = 2(r-1)v_i - (\mu_{i^*} + \delta_{i^*} - 2j^*) \tag{7.54}$$

Dividing through by $2v_i$ (notice $v_i = v_{i^*}$) we get (7.45). ■

Arguing as in the proof of Corollary 7.6 we get

Corollary 7.9. *We have*

$$D_{\overline{e}} f_e(s) = s \prod_{(i,j)\in\Xi} \big(r - C_{i,j}(\mu)\big) \tag{7.55}$$

We can now compute the scalar R from (7.3); we already computed X in (7.14). Starting with (7.18) and plugging in (7.26), (7.37), (7.43) and (7.55) we get

$$R = \frac{\prod C_{i,j}(\mu)}{(r-1)r} - \frac{\prod (C_{i,j}(\mu)+1)}{r(r+1)} - \frac{\prod (r-1-C_{i,j}(\mu))}{(r-1)r} + \frac{\prod (r-C_{i,j}(\mu))}{r(r+1)} \tag{7.56}$$

This is only valid when $r \neq 1$ (as we know $r > 0$).

Fortunately, the expression for R in (7.56) simplifies greatly. We can apply the following formal identity given [B-K4, Lem. 4.8]. Put

$$J(a_i; b) = J(a_0, a_1, a_2, a_3; b) = \frac{a_0 a_1 a_2 a_3}{b(b+1)} \tag{7.57}$$

and $a_m' = b - a_m$ where b, a_0, a_1, a_2, a_3 are five indeterminates. Then

$$\begin{aligned} &J(a_i; b) - J(a_i'; b) - J(a_i + 1; b+1) + J(a_i' + 1; b+1) \\ &= 2b - (a_0 + a_1 + a_2 + a_3) \end{aligned} \tag{7.58}$$

Applying this with $b = r - 1$ and a_1, a_2, a_3, a_4 set equal to the four Capelli multipliers $C_{i,j}(\mu)$ (taken in any order), we get

$$R = 2r - 2 - \sum_{(i,j)\in\Xi} C_{i,j}(\mu) \tag{7.59}$$

To prove (6.22), we need to show $R = X$. We compute

$$\begin{aligned}\sum C_{i,j}(\mu) =& \sum_{i=1}^{q}\sum_{j=0}^{v_i-1}\frac{\mu_i+\delta_i-2j}{2v_i} = \sum_{i=1}^{q}\frac{\mu_i+\delta_i-v_i+1}{2}\\ =& z+\frac{m}{2}-2 = 2r-2-X\end{aligned} \tag{7.60}$$

The second to last equality follows as $\sum_{i=1}^{q}\mu_i = 2z$, $\sum_{i=1}^{q}\delta_i = \sum_{n=1}^{l} d_n(q_n-1)q_n/2 = m-q$ and $\sum_{i=1}^{q} v_i = \sum_{n=1}^{\ell} q_n w_n = 4$, while the last follows by (7.14). So $R = X$. Thus (6.22) holds on s if $s \notin H_1$. This proves Theorem 6.3 in all cases where $r = 1$ never occurs in the spectrum of E' on H.

Now suppose that $r = 1$ does occur and $s \in H_1$. Then Proposition 5.5 implies that $r_0 = 1$ so that H_1 is the vacuum space. Thus by E'-degree $D_e s = D_{\overline{e}} s = 0$. Then (7.26) implies that $\prod C_{i,j}(\mu) = 0$ and so $C_{i,j}(\mu) = 0$ for some (i,j). But also (7.56) collapses to

$$\begin{aligned}R =& \frac{1}{2}\prod(1-C_{i,j}(\mu)) - \frac{1}{2}\prod(C_{i,j}(\mu)+1)\\ =& -\sum_{(i,j)} C_{i,j}(\mu) - \sum_{(i,j)\neq(i',j')\neq(i'',j'')} C_{i,j}(\mu)C_{i',j'}(\mu)C_{i'',j''}(\mu)\end{aligned} \tag{7.61}$$

But (7.60) gives $\sum C_{i,j}(\mu) = -X$. Hence $R = X$ if and only if the third elementary symmetric function of the four numbers $C_{i,j}(\mu)$ is zero. But we already know at least one $C_{i,j}(\mu)$ vanishes. So $R = X$ if and only if at least two of the four numbers $C_{i,j}(\mu)$ are zero.

At this point we observe that $r = r_0 = 1$ implies something very particular about the form of s: in our normal form $s = f_0^p \hat{N}^{\mathbf{t}}\sqrt{\Lambda}$ we have

$$N^{\mathbf{t}} = N_{q_1}^{w_1-u_1} N_{q_1+q_2}^{w_2-u_2} \cdots N_{q_1+\cdots+q_\ell}^{w_\ell-u_\ell} \tag{7.62}$$

where $0 \le u_n \le w_n$. Now it follows from (7.27) that the list of four numbers $C_{i,j}(\mu)$ has at least ℓ zeroes. Thus we are left with the case where $\ell = 1$.

Suppose $\ell = 1$. Then $m \le 4$. To see this, we consider a highest weight vector $s_1 \in H_1$ for the $\mathfrak{k}$-action. Then $s_1 = f_0^{-j}\sqrt{\Lambda}$ for some $j \in \frac{1}{2}\mathbb{Z}$. The eigenvalue of $\mathcal{L}_{\eta^h}$ on s_1 is $-2j+(m+2)/2$ and must be non-negative. The eigenvalue of E' on s_1 is $-j+(m+1)/2$ and is equal to 1. But then $(m+2)/4 \ge j = (m+1)/2 - 1$ and so $4 \ge m$.

Now looking at Table 4.6, we see that $\ell = 1$ and $m \le 4$ only if $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}(p,\mathbb{R})$ where $p = 6, 5$, or 3. We rule out $p = 5$ because of Proposition 5.7(ii). For $p = 6$ we have $r_0 = 1$ when $s_0^2 = \Lambda_0$ and $H_1 \simeq \mathbb{C}$. Then $s = s_0$ and $\mu = (\mu_1,\mu_2) = (0,0)$. The multipliers $C_{i,j}(\mu)$ are $\frac{\mu_1+p-4}{2}, \frac{\mu_2}{2}, \frac{\mu_1+p-6}{2}, \frac{\mu_2-2}{2}$ and this list has two zeroes as required. For $p = 3$, we have $r_0 = 1$ when $s_0^2 = f_0\Lambda_0$ and $H_1 \simeq S^3\mathbb{C}^2$. Then s is one of four vectors with $\mu = (\mu_1)$ where $\mu_1 = 0, 2, 4$, or 6. The multipliers $C_{i,j}(\mu)$ are $\frac{\mu_1}{4}, \frac{\mu_1-2}{4}, \frac{\mu_1-4}{4}, \frac{\mu_1-6}{4}$ and so we never get two zeroes in this list. Thus this one case fails to produce a representation. This concludes the proof of Theorem 6.3.

Next we finish the proof of Theorem 6.8. We started this in §6, and left off at (6.32) where we needed to compute the numbers γ_k defined by $T_{\overline{e}}(f_0^k s_0) = \gamma_k f_0^{k-1} s_0$. But now we

can compute the γ_k because of Proposition 7.8. Indeed, let $s = f_0^k s_0$; then $r = r_0 + k$ and $\mu = \mathbf{deg}(s) = 0$. Now (7.4.3) gives

$$T_{\overline{e}}(f_0^k s_0) = f_0^{k-1} s_0 \frac{\prod_{(i,j)\in\Xi}\big(r_0 + k - 1 - C_{i,j}(0)\big)}{(r_0+k-1)(r_0+k)} \tag{7.63}$$

We can simplify the factor γ_k appearing in (7.63) by computing the four numbers $C_{i,j}(0)$. We have $T_e(s_0) = T_{\overline{e}}(s_0) = 0$ since s_0 is a vacuum vector. Hence, if $r_0 \neq 1$ then the list of four multipliers $C_{i,j}(0)$ contains zero and $r_0 - 1$. Moreover we just showed that if $r_0 = 1$ then the list contains zero with multiplicity at least two. Thus, regardless of the value of r_0, we can write the list of four multipliers $C_{i,j}(0)$ as $0, r_0-1, r_0-a, r_0-b$ where a and b are unknown. Then (7.63) gives

$$\gamma_k = \frac{k(k-1+a)(k-1+b)}{(r_0+k)} \tag{7.64}$$

Consequently

$$\gamma_1 \cdots \gamma_n = \frac{n!(a)_n(b)_n}{(r_0+1)_n} \tag{7.65}$$

Because of (6.32) this gives (6.30). In fact we have gotten the more precise information

Proposition 7.10. *We have equalities of multi-sets:*

$$\{C_{i,j}(0)\}_{(i,j)\in\Xi} = \left\{\frac{\delta_i - 2j}{2v_i}\right\}_{(i,j)\in\Xi} = \{0, r_0-1, r_0-a, r_0-b\} \tag{7.66}$$

In this way P and r_0 determine uniquely the numbers a and b appearing in Theorem 6.8.

Notice that the choice of $\mathfrak{g}_{\mathbb{R}}$ determines P while the choice of $\mathbf{N}^{\frac{1}{2}}$ determines r_0.

Proposition 7.10 implies $\sum C_{i,j}(0) = 3r_0 - 1 - a - b$. But also $\sum C_{i,j}(0) = 2r_0 - 2 - X_0$ by (7.60). Comparing, we get (6.31).

Finally we can use Proposition 7.10 to compute the numbers a and b in Table 6.9. We then observe that a and b are always positive. But we also have a nice theoretical proof of the positivity.

By (7.66), the four numbers $r_0 - C_{i,j}(0)$ are $1, r_0, a, b$. Using first (7.45) and then (7.27) we can write

$$r_0 - C_{i,j}(0) = 1 + C_{i^*,j^*}(\alpha) = \frac{\alpha_{i^*} + \delta_{i^*} + 2v_{i^*} - 2j^*}{2v_{i^*}} \tag{7.67}$$

where $\alpha = 0^*$. The last expression in (7.67) is positive since $\alpha_{i^*}, \delta_{i^*} \geq 0$ and $v_{i^*} > j^*$. This concludes the proof of Theorem 6.8.

§8. The Reproducing Kernel of $\mathcal{H}$.

The aim of this section is to show that the Hilbert spaces $\mathcal{H}$ carrying the unitary irreducible representations constructed in §6 each admit a reproducing kernel $\mathcal{K}$ and $\mathcal{K}$ is a holomorphic half-form on $Y \times \overline{Y}$. It follows then that $\mathcal{H}$ consists entirely of holomorphic half-forms on Y.

We work in the setting of Theorems 6.3, 6.6 and 6.8. (So the one case $G_{\mathbb{R}} = SL(3, \mathbb{R})$, $r_0 = 1$, $H_1 \simeq \mathbb{C}^4$ is excluded as this case did not quantize.)

To begin with, we explain how the notion of reproducing kernel applies here. Our Hilbert space $\mathcal{H}$ is the completion of $H = \Gamma(Y, \mathbf{N}^{\frac{1}{2}})$. Therefore, using the grading (5.15) of H, we may regard a section $s \in \mathcal{H}$ as a formal sum

$$s = \sum_{n \in \mathbb{Z}_+} s_n \tag{8.1}$$

where $s_n \in H_{r_0+n}$. Then s is a holomorphic section of $\mathbf{N}^{\frac{1}{2}}$ if and only if the series in (8.1) converges locally uniformly.

The complex conjugate space $\overline{H}$ identifies naturally with the space $\Gamma(\overline{Y}, \overline{\mathbf{N}}^{\frac{1}{2}})$ of algebraic holomorphic sections of $\overline{\mathbf{N}}^{\frac{1}{2}}$ over $\overline{Y}$. Here $\overline{\mathbf{N}}^{\frac{1}{2}}$ is the complex conjugate line over the complex conjugate algebraic manifold $\overline{Y}$. So we get an identification

$$H \otimes \overline{H} = \Gamma(Y \times \overline{Y}, \mathbf{N}^{\frac{1}{2}} \otimes \overline{\mathbf{N}}^{\frac{1}{2}}) \tag{8.2}$$

A *reproducing kernel* for $\mathcal{H}$ is a section $\mathcal{K}$ of $\mathbf{N}^{\frac{1}{2}} \otimes \overline{\mathbf{N}}^{\frac{1}{2}}$ over $Y \times \overline{Y}$ such that for each $v \in Y$, the formula

$$\mathcal{K}_v(u) = \mathcal{K}(u, \overline{v}) \tag{8.3}$$

defines a section $\mathcal{K}_v \in \mathcal{H} \otimes \overline{\mathbf{N}_v^{\frac{1}{2}}}$ and we have the "reproducing" property for all $s \in \mathcal{H}$

$$s(v) = \langle s | \mathcal{K}_v \rangle \tag{8.4}$$

This makes sense as both sides of (8.4) define vectors in $\mathbf{N}_v^{\frac{1}{2}}$.

$\mathcal{H}$ admits a reproducing kernel if and only if the evaluation map $s \mapsto s(v)$ is continuous on $\mathcal{H}$ for every point $v \in Y$. A reproducing kernel on $\mathcal{H}$, if it exists, is unique and is computed by

$$\mathcal{K} = \sum_k g_k \otimes \overline{g_k} \tag{8.5}$$

where $\{g_k\}$ is any orthonormal basis of $\mathcal{H}$. See , e.g., [F-K, IX,§2] for the case of Hilbert spaces of holomorphic functions.

Each space H_{r_0+n}, $n \in \mathbb{Z}_+$, is finite-dimensional and so admits a reproducing kernel $\Pi_n \in \Gamma(Y \times \overline{Y}, \mathbf{N}^{\frac{1}{2}} \otimes \overline{\mathbf{N}^{\frac{1}{2}}})$. The reproducing kernel $\mathcal{K}$ of $\mathcal{H}$ exists if and only if $\mathcal{K} = \sum_{n \in \mathbb{Z}_+} \Pi_n$, i.e., if and only if the series $\sum_{n \in \mathbb{Z}_+} \Pi_n$ converges.

We have a K-invariant function $T \in R(Y \times \overline{Y})$ defined by

$$T(u, \overline{v}) = (u, \overline{v})_{\mathfrak{g}} \tag{8.6}$$

Then $u \mapsto T(u, \overline{u})$ is a positive real function on Y.

Theorem 8.1. *For any orthonormal basis $\{g_k\}$ of H, the series in* (8.5) *converges locally uniformly and moreover we have the formula*

$$\mathcal{K} = {}_1F_2(r_0 + 1; a, b; T)\Pi_0 \tag{8.7}$$

where a and b are as in Theorem 6.8 *and Π_0 is the reproducing kernel of H_{r_0}. Consequently, $\mathcal{K}$ is a holomorphic section*

$$\mathcal{K} \in \Gamma^{hol}(Y \times \overline{Y}, \mathbf{N}^{\frac{1}{2}} \otimes \overline{\mathbf{N}}^{\frac{1}{2}}) \tag{8.8}$$

Proof. For each n, Π_n is a K-invariant section of $\mathbf{N}^{\frac{1}{2}} \otimes \overline{\mathbf{N}}^{\frac{1}{2}}$. This follows as the Hermitian inner product on H_{r_0+n} is $K_{\mathbb{R}}$-invariant. Hence the quotient Π_n/Π_0 is a K-invariant rational function on $Y \times \overline{Y}$. We have natural actions of $K \times \mathbb{C}^*$ on Y and $\overline{Y}$ where $\mathbb{C}^*$ acts by the Euler scaling action.

Lemma 8.2. *The product action of $K \times \mathbb{C}^*$ on the variety $Y \times \overline{Y}$ has a unique Zariski dense orbit W. The function T separates the K-orbits in W.*

Moreover any K-invariant rational function on $Y \times \overline{Y}$ is a polynomial in T and T^{-1}.

Proof. The isotropy group of K at $(e, \overline{e})$ is $K^e \cap K^{\overline{e}} = K^{\mathfrak{s}} = K_0'$. So the K-orbit of $(e, \overline{e})$ is isomorphic to K/K_0' and hence has codimension 1 in $Y \times \overline{Y}$ by Theorem 4.1 since $\dim_{\mathbb{C}} Y \times \overline{Y} = 2\dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} O$. The $\mathbb{C}^*$-orbit of $(e, \overline{e})$ and the K-orbit of $(e, \overline{e})$ meet in exactly two points, namely $\pm(e, \overline{e})$. This follows from the easy fact that $(a \cdot e, a \cdot \overline{e}) = (te, t\overline{e})$ if and only if $a \in K^h$ and $\chi(a) = t = \chi(a)^{-1}$.

In particular then the $\mathbb{C}^*$-orbit and the K-orbit are transverse at $(e, \overline{e})$. So by dimension, the orbit W of $(e, \overline{e})$ under $K \times \mathbb{C}^*$ is Zariski dense in $Y \times \overline{Y}$.

Now the (punctured) line $\{(te, t\overline{e}) \,|\, t \in \mathbb{C}^*\}$ meets all the K-orbits in W and the function t^2 separates out the points lying in different K-orbits. But the function T is K-invariant and satisfies $T(te, t\overline{e}) = t^2$. So T separates the K-orbits and the last assertion of the Lemma follows easily. ■

Lemma 8.2 implies that Π_n/Π_0 is a polynomial in T and T^{-1}. But also Π_n/Π_0 is bihomogeneous of degree (n, n) under the scaling action of $\mathbb{C}^* \times \mathbb{C}^*$ on $Y \times \overline{Y}$. Since T is bihomogeneous of degree $(1, 1)$, it follows by bihomogeneity that

$$\Pi_n = p_n T^n \Pi_0 \tag{8.9}$$

for some scalar $p_n \in \mathbb{C}^*$.

Our problem now is to compute the scalars p_n. We will do this by writing out the "leading terms" of Π_n, T^n and Π_0. We formulate a notion of leading term in the following way. Suppose V is a highest weight representation of K of weight κ and $S \in V \otimes \overline{V}$ is K-invariant. Then we can write S as a sum of weight vectors S_α where the weight of each S_α is of the form $(\alpha, -\alpha)$. Then we call the term S_κ of weight $(\kappa, -\kappa)$ the *leading term*.

We can identify $R(Y \times \overline{Y}) = R(Y) \otimes R(\overline{Y})$ and then we have the expansion

$$T = \sum_{k=1}^{r} f_{u_k} \otimes \overline{f_{u_k}} \tag{8.10}$$

where $u_1, \ldots, u_r$ is any basis of $\mathfrak{p}$ which is orthonormal with respect to the Hermitian inner product on $\mathfrak{p}$ given by $(u_i|u_j) = (u_i, \overline{u}_j)_{\mathfrak{g}}$. Choosing $u_1, \ldots, u_r$ to be an orthonormal basis by weight vectors, we find (recall $f_e = f_0$ and $(e, \overline{e})_{\mathfrak{g}} = 1$)

$$\text{leading term of } T = f_0 \otimes \overline{f_0} \tag{8.11}$$

and also

$$\text{leading term of } T^n = f_0^n \otimes \overline{f_0^n} \tag{8.12}$$

Next we choose an orthonormal basis on H_{r_0+n} consisting of weight vectors. This basis then contains the highest weight vector $f_0^n s_0 / \|f_0^n s_0\|$ and we find

$$\text{leading term of } \Pi_n = \frac{f_0^n s_0 \otimes \overline{f_0^n s_0}}{\|f_0^n s_0\|^2} \tag{8.13}$$

The leading term of a product is the product of the leading terms, and so equating leading terms in (8.9) we get

$$\frac{f_0^n s_0 \otimes \overline{f_0^n s_0}}{\|f_0^n s_0\|^2} = p_n \left(f_0^n \otimes \overline{f_0^n} \right) (s_0 \otimes \overline{s_0}) \tag{8.14}$$

since $\|s_0\| = 1$. So using (6.30) we find

$$p_n = \frac{1}{\|f_0^n s_0\|^2} = \frac{(r_0+1)_n}{n!(a)_n(b)_n} \tag{8.15}$$

Thus

$$\sum_{n \in \mathbb{Z}_+} \Pi_n = \sum_{n \in \mathbb{Z}_+} \frac{(r_0+1)_n}{n!(a)_n(b)_n} T^n \Pi_0 = {}_1F_2(r_0+1; a, b; T)\Pi_0 \tag{8.16}$$

This proves (8.7). The hypergeometric series here has infinite radius of convergence, and so ${}_1F_2(r_0+1; a, b; T)$ defines a holomorphic function on $Y \times \overline{Y}$. Thus $\mathcal{K}$ is a holomorphic section over $Y \times \overline{Y}$. This concludes the proof of Theorem 8.1. ■

Theorem 8.1 easily gives

Corollary 8.3. *$\mathcal{H}$ consists entirely of holomorphic sections of $\mathbf{N}^{\frac{1}{2}}$ and $\mathcal{K}$ is the reproducing kernel of $\mathcal{H}$.*

§9. Examples of the Quantization.

A feature of our results is that we can construct the representation π in any model of H so long as we are given both the K-module structure and the $R(Y)$-module structure on H. In particular the half-forms can be completely suppressed in the model. We illustrate this by 2 examples. These cases are particularly simple ones where the polynomial P factors in (4.13) into a product of 4 linear terms.

Example 9.1 Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(4,4)$; this is Case (ix) in Table 6.9 with $p = q = 4$. Then $\mathfrak{k} = \mathfrak{sl}(2,\mathbb{C})^{\oplus 4}$. As K-modules we have $H \simeq R(Y) \simeq \oplus_{n\geq 0} S^n(\mathbb{C}^2)^{\otimes 4}$. A model of H is given in the following way. Let S be the polynomial ring in 8 variables $x_{p,i}$ where $p \in \{1,\ldots,4\}$ and $i \in \{1,2\}$. Then H is the subalgebra of S generated by the 16 products $x_{1,i}x_{2,j}x_{3,k}x_{4,l}$ where $i,j,k,l \in \{1,2\}$ so that

$$H = \oplus_{n\geq 0}\mathbb{C}_n[x_{1,1},x_{1,2}]\cdot\mathbb{C}_n[x_{2,1},x_{2,2}]\cdot\mathbb{C}_n[x_{3,1},x_{3,2}]\cdot\mathbb{C}_n[x_{4,1},x_{4,2}] \subset S$$

where $\mathbb{C}_n[u,v]$ is the space of degree n polynomials in u and v. Notice then that H is the space of invariants in S under a scaling action of $\mathbb{C}^*\times\mathbb{C}^*\times\mathbb{C}^*$. Let β be the differential operator on S given by

$$\beta = x_{1,1}\frac{\partial}{\partial x_{1,1}} + x_{1,2}\frac{\partial}{\partial x_{1,2}} + 1$$

Then the following 28 pseudo-differential operators on S preserve H and satisfy the bracket relations of $\mathfrak{so}(8,\mathbb{C})$. I.e., these 28 operators form a basis of a complex Lie algebra $\mathfrak{g}$ isomorphic to $\mathfrak{so}(8,\mathbb{C})$.

$$x_{p,1}\frac{\partial}{\partial x_{p,2}}, \qquad x_{p,2}\frac{\partial}{\partial x_{p,1}}, \qquad x_{p,1}\frac{\partial}{\partial x_{p,1}} - x_{p,2}\frac{\partial}{\partial x_{p,2}} \qquad \text{where } p \in \{1,2,3,4\}$$

$$x_{1,i}x_{2,j}x_{3,k}x_{4,l} - \frac{(-1)^{i+j+k+l}}{\beta(\beta+1)}\frac{\partial^4}{\partial x_{1,i'}\partial x_{2,j'}\partial x_{3,k'}\partial x_{4,l'}}$$

$$\text{where } \{i,i'\} = \{j,j'\} = \{k,k'\} = \{l,l'\} = \{1,2\}$$

Then $r_0 = a = b = 1$ in Table 6.9 and so the $\mathfrak{g}_{\mathbb{R}}$-invariant inner product on H satisfies

$$\left\| \frac{x_{p,i}^n}{n!} \right\|^2 = \frac{(1)_n (1)_n}{n!(2)_n} = \frac{1}{(n+1)}$$

where $p \in \{1, \dots, 4\}$ and $i \in \{1, 2\}$. This agrees with the result in [K].

Example 9.2. Let $\mathfrak{g}_{\mathbb{R}}$ be of type G_2; this is Case (viii) in Table 6.9. Let S be the polynomial ring in 4 variables u_1, u_2, x_1, x_2 and let $S' \simeq R(Y)$ be the subalgebra generated by the 8 products $u_i^3 x_j$ and $u_i^2 u_{i'} x_j$, where $\{i, i'\} = \{1, 2\}$ and $j \in \{1, 2\}$. A model of H is the S'-submodule

$$H = \oplus_{n \geq 0} \mathbb{C}_{3n+2}[u_1, u_2] \cdot \mathbb{C}_n[x_1, x_2] \subset S$$

Let β be the differential operator on S given by

$$\beta = x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_2} + 1$$

The following 14 pseudo-differential operators on S preserve H and satisfy the bracket relations of G_2 so that they form a basis of a complex simple Lie algebra of type G_2.

$$u_1 \frac{\partial}{\partial u_2}, \quad u_2 \frac{\partial}{\partial u_1}, \quad u_1 \frac{\partial}{\partial u_1} - u_2 \frac{\partial}{\partial u_2}$$

$$x_1 \frac{\partial}{\partial x_2}, \quad x_2 \frac{\partial}{\partial x_1}, \quad x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}$$

$$u_i^3 x_j - \frac{(-1)^{i+j}}{27\beta(\beta+1)} \frac{\partial^4}{\partial u_{i'}^3 \partial x_{j'}} \qquad \text{where } \{i, i'\} = \{j, j'\} = \{1, 2\}$$

$$u_i^2 u_{i'} x_j - \frac{(-1)^{i+j}}{27\beta(\beta+1)} \frac{\partial^4}{\partial u_{i'}^2 \partial u_i \partial x_{j'}} \qquad \text{where } \{i, i'\} = \{j, j'\} = \{1, 2\}$$

Then $r_0 = 1$, $a = 4/3$ and $b = 5/3$ in Table 6.9 so that the $\mathfrak{g}_{\mathbb{R}}$-invariant inner product on H satisfies

$$\left\| \frac{u_i^{3n+2} x_j^n}{n!} \right\|^2 = \frac{(4/3)_n (5/3)_n}{n!(2)_n} = \frac{(3n+3)!}{3^{3n} 3! n! (n+1)! (n+1)!}$$

where $i, j \in \{1, 2\}$.

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